Problem Sheet 7

Please send any comments or corrections to `martin@stats.ox.ac.uk`.

1. Sketch the cumulative distribution function of the following distributions:
   (a) the (discrete) uniform distribution on $\{1, 2, \ldots, n\}$;
   (b) the (continuous) uniform distribution on $[a, b]$;
   (c) the exponential distribution with parameter 1;
   (d) the normal distribution with mean 0 and variance 1.

2. For each case below, does there exist a constant c such that the given function is a probability density function? If so, find c and find the cumulative distribution function. (In each case, the given function is zero outside the interval $(0, 1)$.)
   (a) $f_1(x) = cx$ for $0 < x < 1$.
   (b) $f_2(x) = cx^{-1}$ for $0 < x < 1$.
   (c) $f_3(x) = cx^{-1/2}$ for $0 < x < 1$.
   (d) $f_4(x) = c(4x^3 - x)$ for $0 < x < 1$.

3. Let U be a uniformly distributed random variable on $[0, 1]$. Find
   (a) $\mathbb{E}[U]$ and $\mathrm{var}(U)$
   (b) $\mathbb{P}(U < a | U < b)$ for $0 < a < b < 1$.

4. Let X be exponentially distributed with parameter λ.
   (a) Find $\mathbb{P}(X > x)$
   (b) Find $\mathbb{P}(a \leq X \leq b)$ for $0 < a < b$
   (c) Show that $\mathbb{P}(X > a + x | X > a) = \mathbb{P}(X > x)$ for $a, x > 0$. [This is the *memoryless property* of the exponential distribution (compare to Q3 on Problem Sheet 3).]
   (d) Find $\mathbb{P}\left(\sin X > \frac{1}{2}\right)$
   (e) Let $c > 0$. What is the distribution of the random variable cX? [*Try using part (a).*]
   (f) For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the *ceiling* of x; that is, the smallest integer greater than or equal to x. Show that the discrete random variable $\lceil X \rceil$ has a geometric distribution, and find its parameter. [*Hint: write the event* $\{\lceil X \rceil = k\}$ *as* $\{X \in I\}$ *for some interval* I.]

5. Blood plasma nicotine levels in smokers can be modelled by a normal random variable X with mean 315 and variance 131^2, the units being nanograms per millilitre.

   (a) What is the probability that a randomly chosen smoker has nicotine levels lower than 300?

   (b) What is the probability that a randomly chosen smoker has nicotine levels between 300 and 500?

   (c) If 20 smokers are to be tested what is the probability that at most one has a nicotine level higher than 500?

   [*If* $\Phi(x)$ *is the cumulative distribution function of the standard normal distribution then* $\Phi(-0.115) = 0.454$, $\Phi(1.412) = 0.921$.]

6. The radius of a circle is uniformly distributed on $[0, b]$. Find the cumulative distribution function, the probability density function, the expectation and the variance of the random variable representing the area of the circle.

7. Let X be a continuous random variable taking values in $[a, b]$ with c.d.f. F_X which is strictly increasing on $[a, b]$.

   (a) Show that the random variable $F_X(X)$ has a uniform distribution on $[0, 1]$.

   (b) Let U be a uniform random variable on $[0, 1]$. What is the distribution of the random variable $F_X^{-1}(U)$, where F_X^{-1} is the inverse of F_X?

   (c) Suppose that $U_1, U_2, \ldots, U_n$ are a set of computer-generated pseudo-random numbers (assumed to be drawn from a uniform distribution on $[0, 1]$). How would you use them to simulate a random sample $X_1, X_2, \ldots, X_n$ from the distribution with density

$$f(x) = \mu e^{-\mu x}, \quad x \geq 0?$$