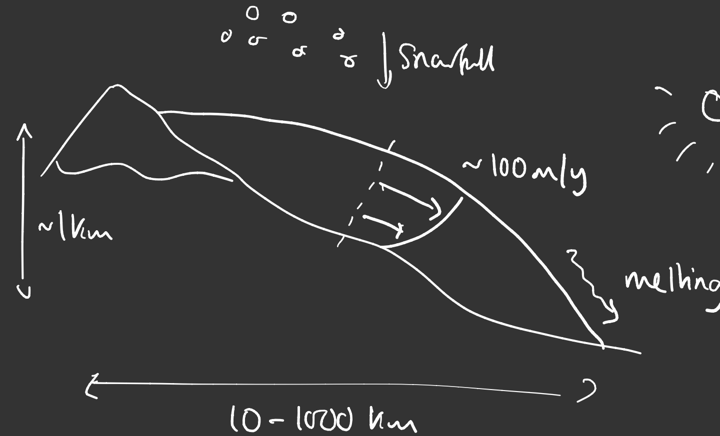
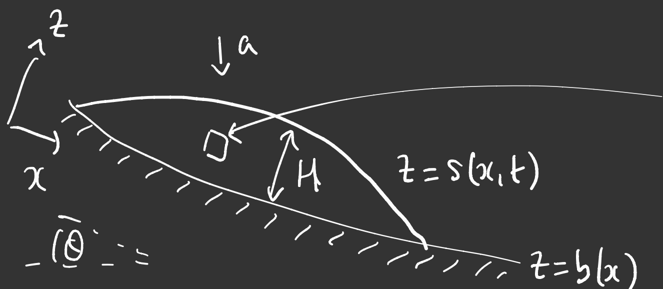
Lecture 12a

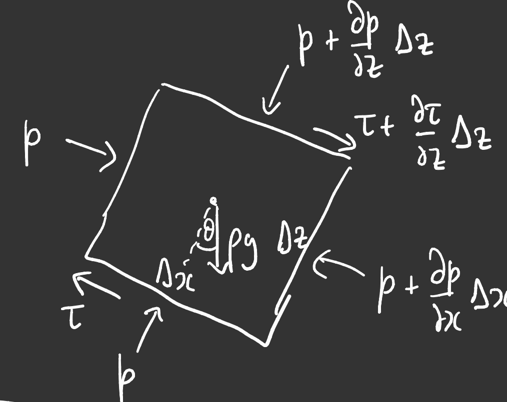
Shallow Ice approximation



Glaciers & ice sheets form through accumulation & compaction of snow at high altitude/latitude and they melt at low altitude/latitude. They move by (i) sliding at their base, and (ii) viscous creep (due to the migration of dislocations in the crystal structure) - we describe this mathematically as a shear-thinning fluid.

Shallow ice approximation





Mass conservation (as for a river): $$\boxed{\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a}$$

where $H = s - b$ is the ice thickness, $q = \int_b^s u \, dz$ is the ice flux, and a is the net accumulation (snowfall − melting).

Force balance: $$0 = -\frac{\partial p}{\partial x}\Delta x \Delta z + \frac{\partial \tau}{\partial z}\Delta z \Delta x + \rho g \sin\theta \, \Delta x \Delta z$$

$$0 = -\frac{\partial p}{\partial z}\Delta z \Delta x - \rho g \cos\theta \, \Delta x \Delta z.$$

$$\Rightarrow \qquad 0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} + \rho g \sin\theta \qquad (1)$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \cos\theta \qquad (2)$$

+ boundary conditions: at $z = s$, $\tau = 0$ (zero shear stress) & $p = 0$ (atmospheric)

$$(2) \Rightarrow \boxed{p = \rho g \cos\theta \, (s - z)}$$

$$(1) \Rightarrow \boxed{\tau = \left(\rho g \sin\theta - \rho g \cos\theta \frac{\partial s}{\partial x}\right)(s - z)}$$

[We have neglected vertical shear stresses & longitudinal stresses, due to the flow being 'shallow', and we've neglected acceleration due to the flow being 'slow' (see online notes)]

Lecture 12b

The final ingredients are (i) a 'constitutive law' or 'flow law' to describe the rheology

For ice, this is Glen's law $\boxed{\frac{1}{2}\frac{\partial u}{\partial z} = A\tau^n}$ (3) where $n \approx 3$, $A \approx 10^{-24}\ \mathrm{Pa}^{-3}\mathrm{s}^{-1}$ (from experiments)

[in reality A depends on temperature, but we ignore that. We could write as $\tau = \eta \frac{\partial u}{\partial z}$ where $\eta = \frac{1}{2A\tau^{n-1}}$ is the effective viscosity. For a Newtonian fluid, $n=1$]

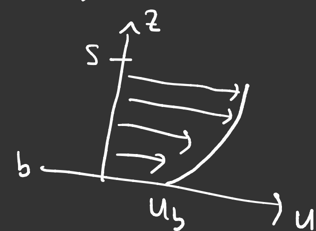
and (ii) a 'sliding law' or a 'friction law' - ice at the bed is usually melting and water facilitates sliding, described by

$\boxed{u_b = C\tau_b^m}$ (4) where $\tau_b = \tau|_{z=b} = \left(\rho g \sin\theta - \rho g \cos\theta \frac{\partial s}{\partial x}\right)H$ is the basal shear stress.

[C may depend on temperature, roughness, and subglacial water pressure, but we treat it as constant here]

Inserting our expression for τ into (3), and integrate subject to (4) (ie. $u = u_b$ at $z = b$)

$$\Rightarrow \quad u = 2A(\rho g)^n \left(\sin\theta - \cos\theta \frac{\partial s}{\partial x}\right)^n \left[\frac{(s-b)^{n+1} - (s-z)^{n+1}}{n+1} \right] + C(\rho g)^m \left(\sin\theta - \cos\theta \frac{\partial s}{\partial x}\right)^m (s-b)^m$$



$(H = s - b)$

Hence the ice flux q is given by

$$\boxed{q = \int_b^s u \, dz = \frac{2A(\rho g \sin\theta)^n}{n+2}\left(1 - \cot\theta \frac{\partial s}{\partial x}\right)^n H^{n+2} + C(\rho g \sin\theta)^m \left(1 - \cot\theta \frac{\partial s}{\partial x}\right) H^{m+1}}$$

(this assumes $\frac{\partial s}{\partial x} < \tan\theta$, otherwise we need to be more careful with signs)

We can now combine our expression for q with conservation of mass $\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a$.

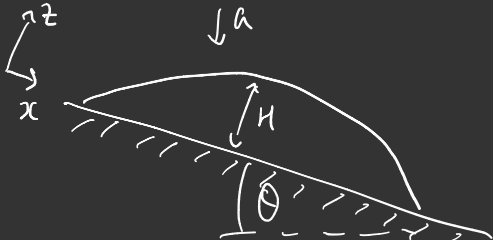
eg. for a glacier on a uniform sloping bed $(b = 0$, so $s = H)$ and with no sliding $(C = 0)$, we find

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\frac{2A(\rho g \sin\theta)^n}{n+2}\left(1 - \cot\theta \frac{\partial H}{\partial x}\right)^n H^{n+2}\right] = a}$$

This is a nonlinear diffusion equation for the ice thickness H.

Lecture 13a

Mountain glaciers



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\frac{2A(\rho g \sin\theta)^n}{n+2} \left(1 - \cot\theta \frac{\partial H}{\partial x}\right)^n H^{n+2} \right] = a$$

Non-dimensionalisation: given length scale $[x]$, and an accumulation scale $[a]$, we choose

$$[H]^{n+2} = \frac{[x][a]}{2A(\rho g \sin\theta)^n}, \quad [t] = \frac{[H]}{[a]}$$

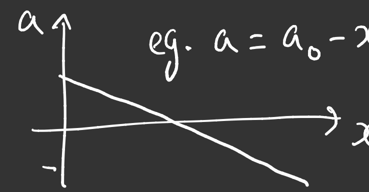
$$\Rightarrow \boxed{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\frac{H^{n+2}}{n+2}\left(1 - \mu \frac{\partial H}{\partial x}\right)^n \right] = a}$$

where $\mu = \dfrac{\cot\theta\,[H]}{[x]}$

Typical values: $n = 3$, $A = 10^{-24}\ \mathrm{Pa^{-3}\,s^{-1}}$, $\rho = 916\ \mathrm{kg\,m^{-3}}$, $g = 9.8\ \mathrm{m\,s^{-2}}$, $\sin\theta = 0.1$

$[x] = 10\ \mathrm{km}$ $[a] = 1\ \mathrm{m/y}$ $\Rightarrow$ $[H] \hat{=} 200\ \mathrm{m}$, $[t] \hat{=} 200\ \mathrm{y}$, $\mu \hat{=} 0.2$

Approximate $\mu << 1$. Then $\dfrac{\partial H}{\partial t} + H^{n+1}\dfrac{\partial H}{\partial x} = a(x)$



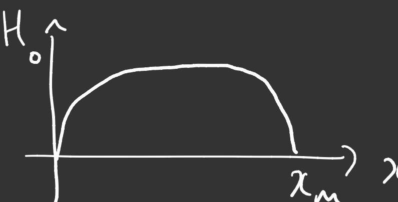
eg. $a = a_0 - x$

Solve with $H=0$ at $x=0$, the head of the glacier.

The glacier ends at the terminus $x = x_m(t)$ – a free boundary.

For the steady state, $\dfrac{\partial}{\partial x}\left[\dfrac{H^{n+2}}{n+2}\right] = a(x)$ with $H=0$ at $x=0$

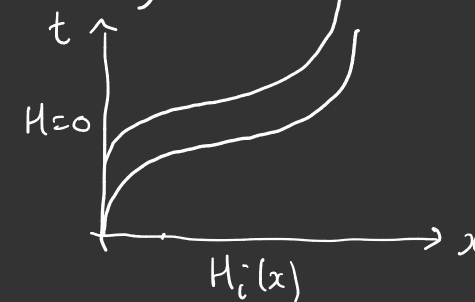
$$\Rightarrow \boxed{H_0 = \left[(n+2)\int_0^x a(\hat{x})\,d\hat{x}\right]^{\frac{1}{n+2}}}$$



The terminus position x_m is determined from where the ice flux (and hence H) go to zero. ie. where $\displaystyle\int_0^{x_m} a(\hat{x})\,d\hat{x} = 0$.

eg. if $a = a_0 - x$, then $H_0 = \left[(n+2)\,x\left(a_0 - \tfrac{1}{2}x\right)\right]^{\frac{1}{n+2}}$ with $x_m = 2a_0$

For a given initial condition $H = H_i(x)$ at $t=0$. We solve using characteristics,



$$\dot t = 1, \quad \dot x = H^{n+1}, \quad \dot H = a(x)$$

with, at $t=0$, $x = \sigma$, $H = H_i(\sigma)$

and at $t=\tau$, $x = 0$, $H = 0$

Note that along each characteristic $\dfrac{dH}{dx} = \dfrac{\dot H}{\dot x} = \dfrac{a}{H^{n+1}} \Rightarrow \boxed{H^{n+1}\dfrac{dH}{dx} = a}$

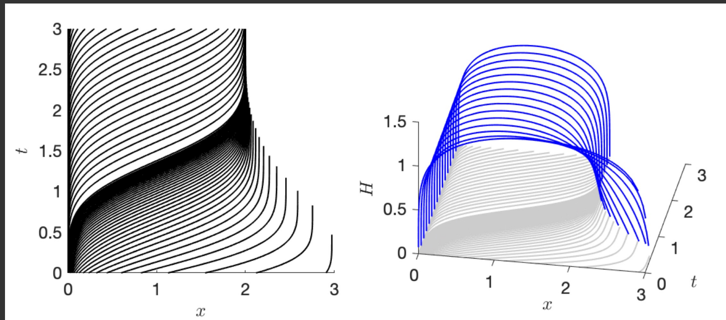
For the characteristics starting from $x=0$, we have $\dfrac{H^{n+2}}{n+2} = \displaystyle\int_0^x a(\hat x)\,d\hat x$ ie $H = H_0(x)$, the steady state.

The characteristics are $\dot x = H_0^{n+1} \Rightarrow t = \displaystyle\int_0^x \frac{d\hat x}{H_0(\hat x)^{n+1}} + \tau$

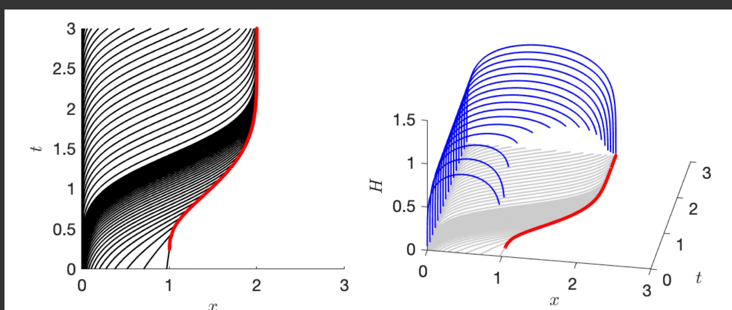
For the characteristics starting from $t=0$, we have $\dfrac{H^{n+2}}{n+2} = \displaystyle\int_\sigma^x a(\hat x)\,d\hat x + \frac{H_i(\sigma)^{n+2}}{n+2}$

The characteristics have $\dot x = H(x,\sigma)^{n+1}$ $\Rightarrow t = \displaystyle\int_\sigma^x \frac{d\hat x}{H(\hat x,\sigma)^{n+1}}$ $\leftarrow$ determines $\sigma(x,t)$

Lecture 13b



← starting with too large a glacier



← starting with too small a glacier

If H is close to the steady state $H_0(x)$, it is easier to linearise the model.

Write $H = H_0(x) + H_1(x,t)$, and suppose $H_1 \ll H_0$ $\left(\text{in } \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(\frac{H^{n+2}}{n+2}\right) = a \right)$

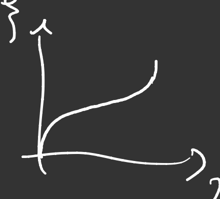
$$\Rightarrow \quad \frac{\partial H_1}{\partial t} + \frac{\partial}{\partial x}\left[H_0^{n+1} H_1 \right] = 0$$

[note this is not a constant coefficient equation, since $H_0 = H_0(x)$]

multiply by H_0^{n+1} $\Rightarrow$

$$\frac{\partial}{\partial t}\left(H_0^{n+1}H_1\right) + \underbrace{H_0^{n+1}\frac{\partial}{\partial x}}_{\frac{\partial}{\partial \xi}}\left(H_0^{n+1}H_1\right) = 0$$

if $\xi = \int_0^x \frac{d\hat{x}}{H_0(\hat{x})^{n+1}}$



i.e. $\frac{\partial}{\partial t}\left(H_0^{n+1}H_1\right) + \frac{\partial}{\partial \xi}\left(H_0^{n+1}H_1\right) = 0$

which has solution $H_1 = \frac{\phi(\xi - t)}{H_0^{n+1}}$ where ϕ is determined from initial conditions.

Perturbations travel down glacier with variable amplitude. The linearisation breaks down near the terminus.

An alternative method - appropriate for small wave length perturbations - is to approximate the characteristic equations $\dot{t} = 1$ $\quad \dot{x} = H^{n+1}$ $\quad \dot{H} = a$

$$\approx H_0^{n+1} \qquad \dot{H}_0 = a \ \& \ \dot{H}_1 = 0$$

This is equivalent to solving the PDE $\quad \dfrac{\partial H_1}{\partial t} + H_0^{n+1} \dfrac{\partial H_1}{\partial x} = 0$

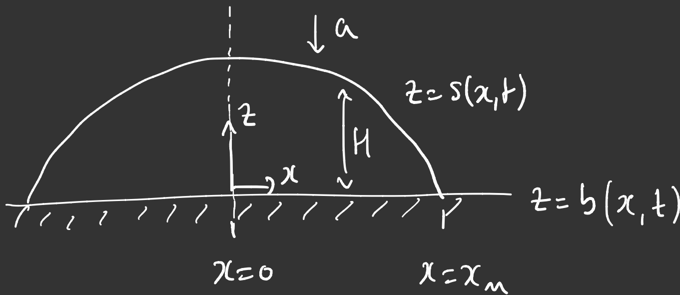
(compared to before, we're missing a term $H_1 \dfrac{\partial}{\partial x}(H_0^{n+1})$ - ok to do this if H_1 varies on short wavelengths)

This has solution $H_1 = \phi(\xi - t)$ where ξ is defined as before.

This method can be used to consider the slight annual growth / shrinkage of a glacier (see problem sheet)

Lecture 14a

Ice Sheets



Mass conservation $$\boxed{\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a}$$

(we consider $x > 0$, where we assume $\frac{\partial s}{\partial x} < 0$)

Force balance: $$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} \qquad \Rightarrow \quad \tau = -\rho g \frac{\partial s}{\partial x}(s-z)$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \qquad \Rightarrow \quad p = \rho g (s-z)$$

Flow law: $$\frac{\partial u}{\partial z} = 2A\tau^n$$

Sliding law: $$u_b = C\tau_b^m = \left(\frac{\tau_b}{c}\right)^m$$

$$\Rightarrow u = \frac{2A(\rho g)^n}{n+1}\left(-\frac{\partial s}{\partial x}\right)^n\left[(s-b)^{n+1} - (s-z)^{n+1}\right] + \left(\frac{\rho g}{c}\right)^m\left(-\frac{\partial s}{\partial x}\right)^m (s-b)^m$$

$$\boxed{q = \int_b^s u\,dz = \frac{2A(\rho g)^n}{n+2}\left(-\frac{\partial s}{\partial x}\right)^n H^{n+2} + \left(\frac{\rho g}{c}\right)^m\left(-\frac{\partial s}{\partial x}\right)^m H^{m+1}}$$

Boundary conditions: $q = 0$ at $x = 0$ (symmetry) and $q = H = 0$ at $x = x_m$.

Suppose the bed is flat ($b=0$) and motion is dominated by sliding, then

$$q = \left(\frac{\rho g}{c}\right)^m \left(-\frac{\partial H}{\partial x}\right)^m H^{m+1} \quad \Rightarrow \quad q^{1/m} = -\frac{\rho g}{c} H^{1+1/m} \frac{\partial H}{\partial x} \qquad (*)$$

For a steady state, with given $a(x)$, we have $q = \int_0^x a(\hat{x})\,d\hat{x}$, and x_m is determined from the constraint $\int_0^{x_m} a(\hat{x})\,d\hat{x} = 0$. Then $(*)$ becomes

$$\frac{\rho g}{c} H^{1+1/m} \frac{\partial H}{\partial x} = -\left(\int_0^x a(\hat{x})\,d\hat{x}\right)^{1/m}$$

and we have $H=0$ at $x = x_m$.

$$\Rightarrow \quad \frac{\rho g}{c} \frac{m}{2m+1} H^{\frac{2m+1}{m}} = \int_x^{x_m} \left(\int_0^{x'} a(\hat{x})\,d\hat{x}\right)^{1/m} dx'$$

$$\Rightarrow \quad \boxed{H = \left[\frac{2m+1}{m} \frac{c}{\rho g} \int_x^{x_m} \left(\int_0^{x'} a(\hat{x})\,d\hat{x}\right)^{1/m} dx'\right]^{\frac{m}{2m+1}}}$$

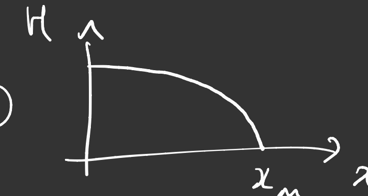
Note that integrating the mass conservation equation over the ice sheet gives a global mass conservation equation,

$$0 = \int_0^{x_m} \frac{\partial q}{\partial x}\, dx = \int_0^{x_m} a - \frac{\partial H}{\partial t}\, dx = \int_0^{x_m} a\, dx - \frac{\partial}{\partial t}\left(\int_0^{x_m} H\, dx\right) \Rightarrow \boxed{\frac{\partial}{\partial t}\left(\int_0^{x_m} H\, dx\right) = \int_0^{x_m} a\, dx} \quad (1)$$

It is helpful to make the 'plastic' approximation $n \to \infty$. (*) in this case becomes

$$c = -\rho g H \frac{\partial H}{\partial x} \quad \text{and we know } H = 0 \text{ at } x = x_m.$$

$$\Rightarrow \quad c(x - x_m) = -\frac{1}{2}\rho g H^2 \quad \Rightarrow \quad \boxed{H = \underbrace{\left(\frac{2c}{\rho g}\right)^{1/2}}_{H_0^{1/2}} (x_m - x)^{1/2}} \quad (2)$$



Lecture 14b

Melt-elevation feedback. We expect a to depend on x (latitude) and H (elevation).

We take $a = a_0 - \mu x + \lambda H$

(note there is an equilibrium altitude $H = \frac{-a_0 + \mu x}{\lambda}$ at which $a = 0$, and above which $a > 0$)

Using ② for the ice thickness, we have

$$\int_0^{x_m} H \, dx = \frac{2}{3} H_0^{1/2} x_m^{3/2} \quad \& \quad \int_0^{x_m} a \, dx = a_0 x_m - \frac{1}{2}\mu x_m^2 + \frac{2}{3}\lambda H_0^{1/2} x_m^{3/2}.$$

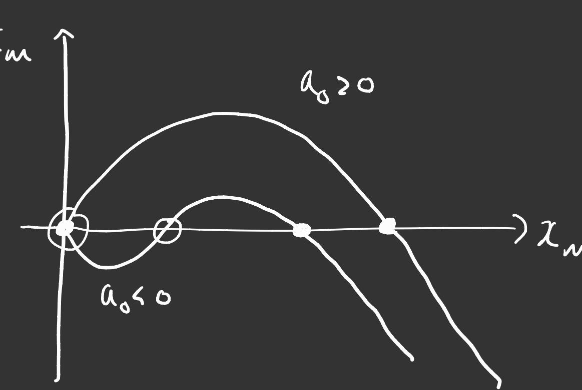
Then mass conservation ① $\Rightarrow$ $H_0^{1/2} x_m^{1/2} \dot{x}_m = a_0 x_m - \frac{1}{2}\mu x_m^2 + \frac{2}{3}\lambda H_0^{1/2} x_m^{3/2}$

$$\Rightarrow \boxed{\dot{x}_m = \frac{x_m^{1/2}}{H_0^{1/2}}\left[a_0 - \frac{1}{2}\mu x_m + \frac{2}{3}\lambda H_0^{1/2} x_m^{1/2}\right]}$$

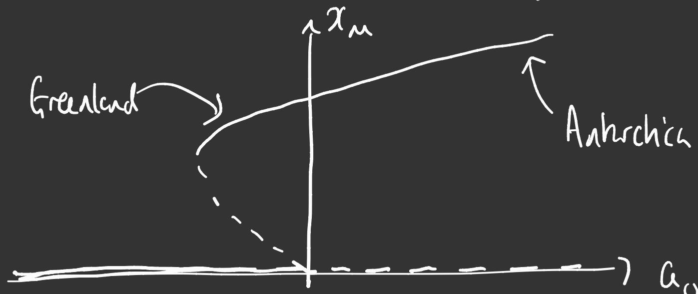
$$\boxed{\dot{x}_m = \frac{x_m^{1/2}}{H_0^{1/2}}\left[a_0 - \tfrac{1}{2}\mu x_m + \tfrac{2}{3}\lambda H_0^{1/2} x_m^{1/2}\right]}$$

$$\underbrace{\phantom{a_0 - \tfrac{1}{2}\mu x_m + \tfrac{2}{3}\lambda H_0^{1/2} x_m^{1/2}}}$$

$$= 0 \text{ at } x_m^{1/2} = \frac{2}{3}\frac{\lambda}{\mu}H_0^{1/2} \pm \left(\left(\frac{2}{3}\frac{\lambda}{\mu}H_0^{1/2}\right)^2 + \frac{2a_0}{\mu}\right)^{1/2}$$



We can construct a bifurcation diagram of steady-state ice-sheet size



There is the possibility of hysteresis as the climate (ie a_0) changes.

The model also shows the timescale on which the ice sheet evolves $t \sim \dfrac{H_0^{1/2} x_m^{1/2}}{a_0}$

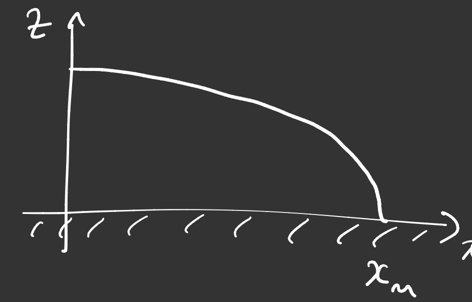
taking $x_m = 500$ km, $H_0 \approx 20$ m, $a_0 \sim 1$ m/y $\Rightarrow t \sim \dfrac{(10^7 \text{ m}^2)^{1/2}}{1 \text{ m/y}} \approx 3000$ y.

Lecture 15a

Marine ice sheets

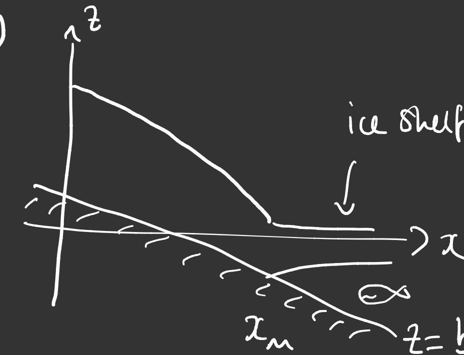
For a land-terminating ice sheet, $q = H = 0$ at $x = x_m$

so steady states have $\int_0^{x_m} a \, dx = 0$.



For Antarctica, $a > 0$ almost everywhere, and ice is lost by flowing into the ocean, and calving icebergs.

Ice sheet becomes afloat at $x = x_m$, where $\boxed{\rho g H = -\rho_w g b}$ (Archimedes). x_m is called the 'grounding line'.



A theory for the ice shelf (see online notes), suggests it is appropriate to apply

$\boxed{q = BH^{\alpha}}$ at $x = x_m$, for some constants $B, \alpha > 0$

The model for a marine ice sheet becomes $\dfrac{\partial H}{\partial t} + \dfrac{\partial q}{\partial x} = a$

with $q = 0$ at $x = 0$

and $q = B H^{\alpha}$ at $x = x_m$

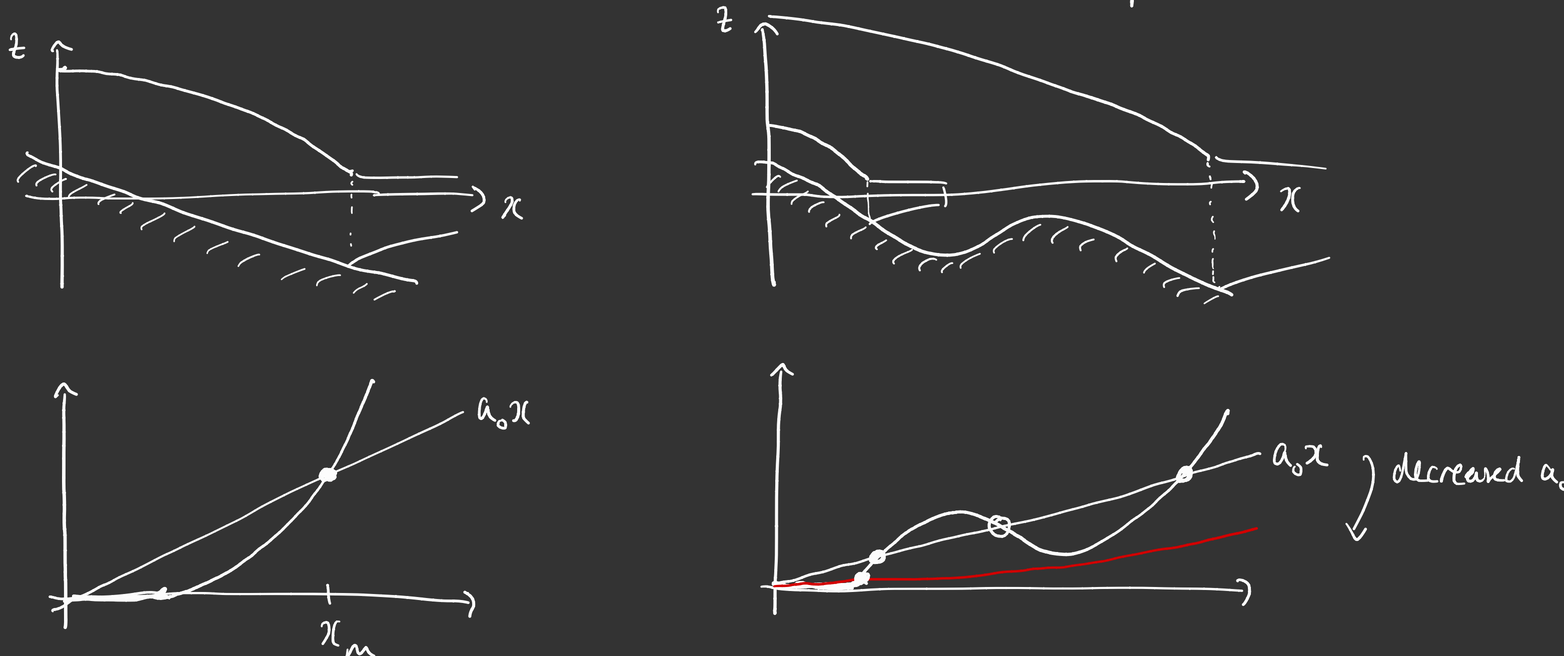
$H = \max\left(0, -\dfrac{\rho_w}{\rho} b\right)$ at $x = x_m$

For a steady state, $$\boxed{\int_0^{x_m} a \, dx = \int_0^{x_m} \frac{\partial q}{\partial x} \, dx = B \max\left(0, -\frac{\rho_w}{\rho} b\right)^{\alpha}}$$

which determines the steady state position of the grounding line x_m.

[We could use the plastic sliding approximation, $-\rho g H\left(\dfrac{\partial H}{\partial x} + \dfrac{\partial b}{\partial x}\right) = c$ to find $H(x)$]

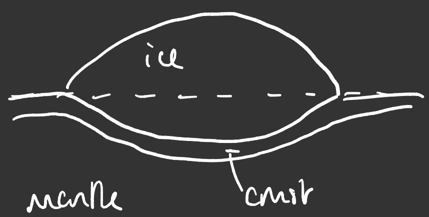
e.g. $a = a_0 > 0$ is constant. We need $\quad a_0 x_m = B \max\left(0, -\frac{\rho_w}{\rho} b(x_m)\right)^{\alpha}$



If bedrock is non-monotonic, there may be multiple steady solutions for x_m, and the possibility of hysteresis as a_0 changes. If steady states disappear, there can be rapid retreat to another steady state (it is believed this may happen to West Antarctica).

Lecture 15b

Isostacy

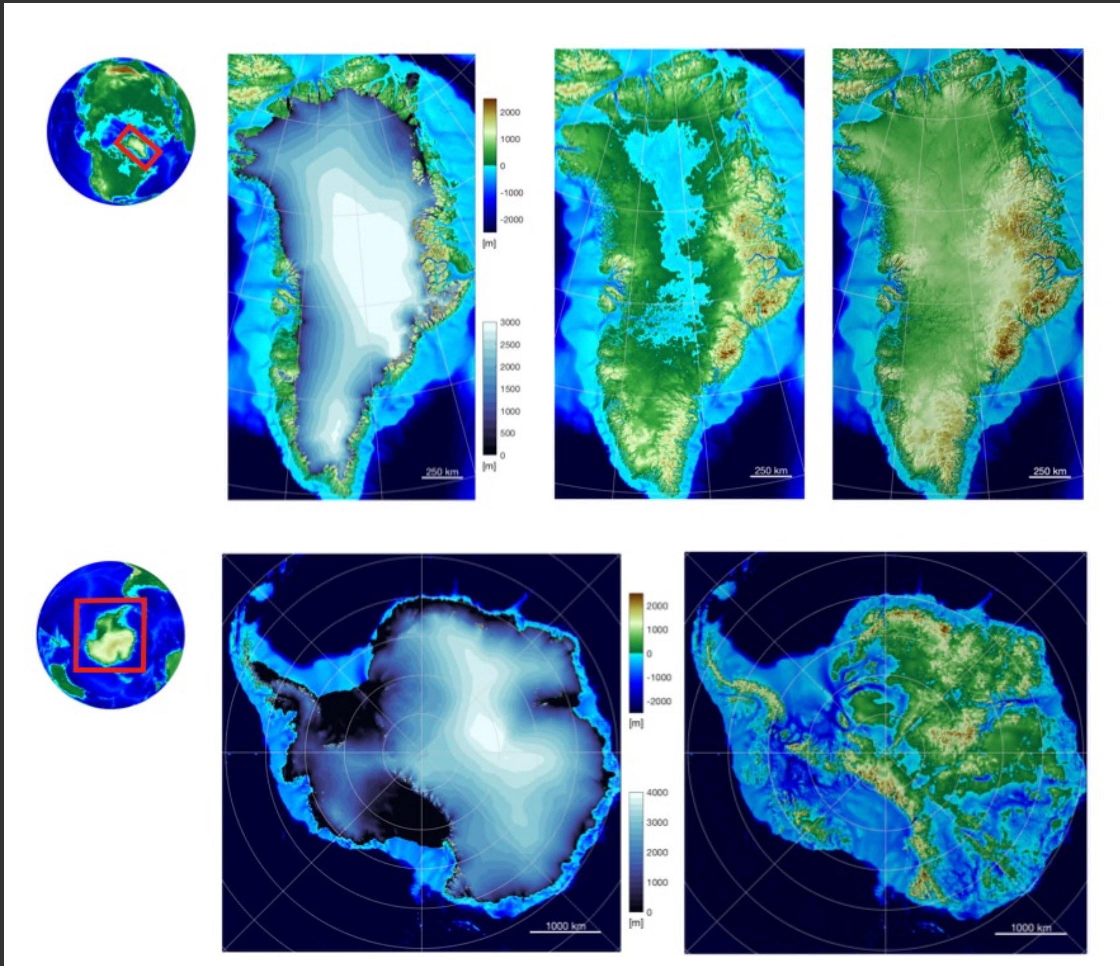


The Earth's mantle is relatively 'fluid', so the crust sinks under the weight of the ice. A simple model is to assume the ice 'floats' on the mantle, using Archimedes' principle.

If the bed is at $z = b_0(x)$ in the absence of ice, then we need

$$\rho g H = \rho_m g (b_0 - b), \quad \text{and hence} \quad b(x,t) = b_0(x) - \frac{\rho}{\rho_m} H(x,t)$$

The ice surface is then $s = b_0 + \left(1 - \rho/\rho_m\right) H \qquad \left(\rho/\rho_m \approx \frac{1}{3}\right)$



Greenland ice sheet

- with the ice stripped away
- with 'isostatic compensation'

Antarctic ice sheet

- much of the ice is grounded on bed well below sea level.

Sea level

Globally-averaged sea level change can be calculated by dividing the volume of water that melts from the ice sheets ($\rho/\rho_w \approx 0.9$ times the volume of ice) by the surface area of the ocean $A_o \approx 362 \times 10^6\ km^2$.

eg. Greenland has $2.9 \times 10^6\ km^3$ ice $\Rightarrow$ 7 m of sea level equivalent.

Antarctica has $26.5 \times 10^6\ km^3$ ice $\Rightarrow$ 67 m, but only volume above flotation counts, so this is 58 m.

Lecture 16a

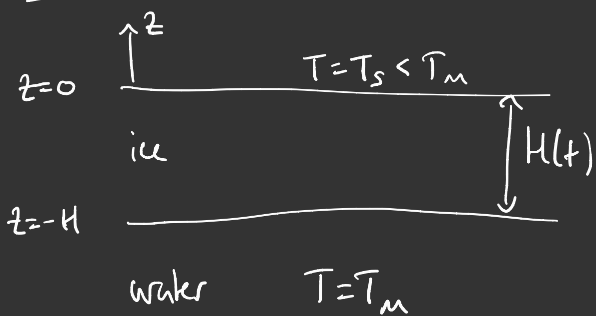
Sea ice

Sea ice forms when the ocean cools and freezes from the surface downwards.

Salinity of ocean water complicates the physics of sea ice formation, so for simplicity we'll consider freezing fresh water.

The simplest model is the <u>Stefan problem</u>

Stefan problem



Water initially at its freezing temperature $T=T_m$ ($=0C$) is subjected to a cold surface temperature $T=T_s<T_m$.

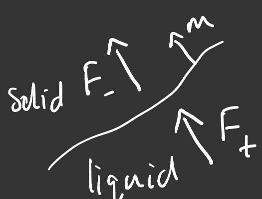
A layer of ice grows downwards from the surface, with thickness $H(t)$.

In the ice, temperature $T(z,t)$ satisfies the heat equation

$$\boxed{\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \qquad \text{with} \quad T=T_s \text{ at } z=0, \quad T=T_m \text{ at } z=-H}$$

We also have $\dfrac{dH}{dt} = -m$, where m is the melt rate (negative for freezing).

At the ice-water interface, the Stefan condition determines m;



$$\rho L m = F_+ - F_-$$
$$= 0 - \left(-k\frac{\partial T}{\partial z}\right)\bigg|_{z=-H}$$

$$\Rightarrow \quad \boxed{\rho L \frac{dH}{dt} = -k\frac{\partial T}{\partial z}\bigg|_{z=-H}}$$

& $H(0)=0$

Non-dimensionalise: $T = T_m + [T]\hat{T}$, $z = [z]\hat{z}$, $H = [z]\hat{H}$, $t = [t]\hat{t}$, & choose

$$[T] = T_m - T_s, \quad [t] = \frac{\rho L [z]^2}{k[T]}.$$

$\Rightarrow$ (dropping hats) $\quad \dfrac{1}{S}\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial z^2}$, with $T = -1$ at $z = 0$, and $T = 0$ at $z = -H(t)$

& $\quad \dfrac{dH}{dt} = -\dfrac{\partial T}{\partial z}\Big|_{z=-H}$ with $H(0) = 0$.

Here $S = \dfrac{L}{c[T]}$ is the <u>Stefan number</u>. For water, $\dfrac{L}{c} \simeq 165\,K$, and $[T] \simeq 10\,K$, so expect <u>$S \gg 1$</u>.

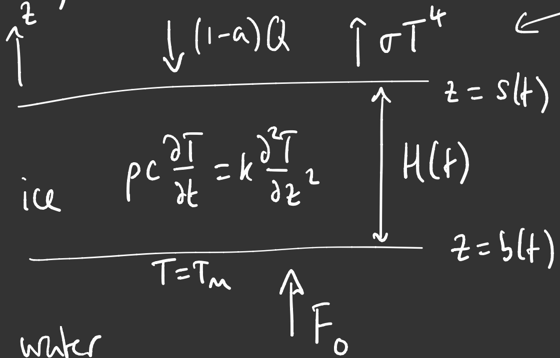
If we take $S \to \infty$, the temperature T is quasi-steady, and is given by $T = -1 - \dfrac{z}{H}$, and the Stefan condition becomes $\dfrac{dH}{dt} = \dfrac{1}{H}$ with $H(0) = 0$ $\Rightarrow$ $\boxed{H = \sqrt{2t}}$

[Note, the full problem (for $O(1)$ S) has a similarity solution (see online notes)]

Lecture 16b

A better model for sea ice formation has modified boundary conditions, and allows for melting as well as freezing.

radiative fluxes (cf. lecture 1)



$$\frac{ds}{dt} = -m_s$$

$$\frac{db}{dt} = m_b$$

$$\Rightarrow \quad \underbrace{\frac{dH}{dt} = -m_s - m_b}_{(1)}$$

At $z = b$, the Stefan condition says $\quad \underbrace{\rho L m_b = F_0 - \left(-k\frac{\partial T}{\partial z}\right)}_{(2)}\Big|_{z=b}$ & $T = T_m$

At $z = s$, either $T \leq T_m$, and $(1-a)Q - \sigma T^4 = k\frac{\partial T}{\partial z}\Big|_{z=s}$. & $m_s = 0$

or $T = T_m$, and $\underbrace{(1-a)Q - \sigma T^4 - k\frac{\partial T}{\partial z}\Big|_{z=s} = \rho L m_s}_{(3)} \geq 0$

Non-dimensionalise, writing $T = T_m + [T]\hat{T}$, then $\sigma T^4 = \sigma T_m^4\left(1 + \frac{[T]}{T_m}\hat{T}\right)^4 \approx \sigma T_m^4 + 4\sigma T_m^3 [T]\hat{T}$

Choose ① $[m_s] = [m_b] = \frac{[z]}{[t]}$, ② $[t] = \frac{\rho L [z]^2}{k[T]}$, ③ $[z] = \frac{k}{4\sigma T_m^3}$.

Also define $S = \frac{L}{c[T]}$, $\hat{F}_0 = \frac{F_0[z]}{k[T]}$, $\hat{Q} = \frac{(1-a)Q - \sigma T_m^4}{4\sigma T_m^3 [T]}$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{1}{S}\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial z^2} \quad \text{for } b < z < s, \text{ with} \\[2ex] \text{at } z=b, \begin{cases} T = 0 \\ m_b = \hat{F}_0 + \dfrac{\partial T}{\partial z} \end{cases}, \text{ and at } z=s \begin{cases} T \le 0, & \hat{Q} - T = \dfrac{\partial T}{\partial z}, \ m_s = 0 \\ T = 0, & \hat{Q} - T - \dfrac{\partial T}{\partial z} = m_s \geq 0 \end{cases} \end{array}}$$

Note $\hat{Q}$ can vary in time, typically < 0 during winter and ≥ 0 during summer (see problem sheet).