# The method of *fang cheng*, an ancient Chinese algorithm to solve simultaneous equations

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When describing history, there is often a rather macabre way of saying that it is written by the victors. Whilst it would be almost impossible for us, mathematicians, to assess the validity of this view, some inklings of that idea could perhaps be found within our own subject as well. The mathematics being taught currently throughout the world at a university level is based heavily on the western perspective. We as students, who learn the subject under this system, would be forgiven for thinking that this is how the subject of mathematics evolved. However, much like the history of everything else, the actual evolution of mathematics was messy, diverse, and far more intriguing that one would imagine. Civilisations throughout the world have developed mathematics in their own ways, and in so many cases, they discovered theorems, methods, and algorithms far earlier than the western mathematicians to whom the results are usually attributed. Unfortunately for many of them, the later prevalence of the western-centric approach in academia meant that those achievements would only fade into obscurity. However, it would be a shame to disregard those contributions, and this article explores one of those 'forgotten' pieces of mathematical treasure.

The algorithm for solving simultaneous equations by row reductions is often known to modern mathematicians as Gaussian Elimination, a topic which is covered in the opening chapters of Prelims Linear Algebra. However, Carl Friedrich Gauss was far from the first to derive such a method: long before his birth in 1777, mathematicians elsewhere in the world were familiar with this algorithm. The earliest of these mathematicians could be found in ancient China, during the waning years of the Eastern Han dynasty, almost 2000 years ago.

In 179 CE, just five years before the Yellow Turban Rebellion (黄巾起义) that sparked the series of wars which lead to the famed Three Kingdom period, the first known references to *The Nine Chapters on the Mathematical Art* (九章算术) were found on two bronze standard measures. The book, although only explicitly mentioned and given a title in 179 CE, was the culmination of the works of many generations of scholars from all the way back into the 10th century BCE, during the early years of the Zhou dynasty.

The *Nine Chapters* deals with a different area of mathematics in each of its chapters, ranging from basic multiplication and division all the way to algebra and geometry. Amongst these, the method of Gaussian elimination, or the method of *fang cheng* (方程/Rectangular Arrays) rather, can be found in the penultimate chapter and the name itself provides insight into how well the ancient Chinese understood the subject. Although *fang cheng* has taken on the meaning of equations or functions in modern-day China, its original concept in the *Nine Chapters* suggested ideas far deeper than a simple arrangement of unknown variables. The word *fang*, in this context, means rectangular, and yet instead of being related to a geometrical concept, it refers to the way equations of unknowns are arranged. This configuration of variables, as we will show below, is very much reminiscent of the modern matrix notation.

The first question of this chapter deals with finding the measure of three types of cereal grains from three simultaneous equations, and the translation is given below. Of course, for the sake of clarity, the rest of this article can still be easily read whilst completely skipping these translated texts (which are written *in italics and a smaller font*), as there will always be a modern restatement of them immediately following each passage.

> *Now there are* $3$ *bundles of top grade cereal,* $2$ *bundles of medium grade cereal and* $1$ *bundle of low grade cereal,* [*which yield*] $39$ dou [*of grains*] *as* shi*;* $2$ *bundles of top grade cereal,* $3$ *bundles of medium grade cereal and* $1$ *bundle of low grade cereal* [*yield*] $34$ dou *as* shi*;* $1$ *bundle of top grade cereal,* $2$ *bundles of medium grade cereal and* $3$ *bundles of low grade cereal* [*yield*] $26$ dou *as* shi*. Find the measure* [*of grains*] *in each bundle of the top, medium and low grade cereals.*[1]

When written out in terms of our modern notation, this is simply the three following equations: $3x + 2y + z = 39$, $2x + 3y + z = 34$ and $x + 2y + 3y = 26$, with $x, y, z$ representing the measures of top, medium and low grade cereal grains, respectively. Solving this should not present too much difficulty, especially when using the method of Gaussian Elimination, as given in the Prelims Linear Algebra lecture notes: we can simply write out this set of equations in terms of a $3 \times 4$ matrix, and apply row reduction.

Most remarkably, the Gaussian method we have learned is almost eerily similar to the way that the solution was presented by the *Nine Chapters*, especially considering that it was written more than 1500 years before the first inklings of its idea appeared in Europe. For modern students of mathematics, it might be confusing to hear why we would consider this similarity notable, and indeed, it should be intuitive to us that mathematics would develop in a method that is logical, and each generation of mathematicians would simply build upon the works of their predecessors and eventually, the subject would grow into the form that we know now. Unfortunately for historians, the development of mathematics can often be far from intuitive, and the works of prominent mathematicians even from as recently as two centuries ago can seem as nonsense to the modern reader, and similarly, a mathematical concept will often go through many iterations before being generally accepted by both the academic community and the general public and evolve into its modern form. Take one subject for example, negative numbers: a concept we take for granted nowadays, appearing in many ancient texts, discussed and used frequently throughout history, and yet its position inside the mathematical canons were so tenuous that up until even the early 19th century, it was sometimes demonised and villainised by some prominent mathematicians. Negative numbers were frequently branded as

[1] Lam and Ang, 2004, 148–150. This, and all succeeding translations, are due to Lam and Ang.

'absurd', 'un-intelligible', 'nonsense', 'jargon', and 'mistaken forms'. Given the tumultuous history of many mathematical concepts, it is truly remarkable how succinctly and elegantly this method was presented. So much so that, if it were not for Occam's razor, one's first immediate reaction upon seeing it could have been that somehow, a mathematics student from our time must have procured a time machine and taken the idea back to the Warring States period.

Going back to the question given in the *Nine Chapters*, the solution is given in the book as follows:

> *Put down* $3$ *bundles of top grade cereal,* $2$ *bundles of medium grade cereal,* $1$ *bundle of low grade cereal and* $39$ dou *as* shi *in a column on the right. Set up the columns in the centre and on the left in the same way as the column on the right. Take the* [*number representing*] *top grade cereal in the right column to multiply all* [*numbers*] *in the central column, and then use* [*the method*] *of direct subtractions (*zhi chu*). Once again multiply* [*the numbers*] *in the next column* [*that is, the left column, by the number representing top grade cereal in the right column*]*, and then use* [*the method of*] *direct subtractions (*zhi chu*). Next multiply all* [*the numbers in*] *the left column by the remaining* [*number representing*] *medium grade cereal in the central column, and then use* [*the method of*] *direct subtractions. The left column has the remaining number* [*representing*] *low grade cereal. The* fa *(divisor) is above and the* shi *(dividend) below; the* shi *here is the* shi *for low grade cereal. To find* [*the measure for*] *medium grade cereal, multiply the* shi *in the central column by the* fa [*of the left column*]*, and subtract the* shi *for low grade cereal. The remainder is divided by the number of bundles of medium grade cereal* [*in the central column*]*, yielding the* shi *for medium grade cereal. To find* [*the measure for*] *top grade cereal, once again multiply the* shi *in the right column by the* fa [*of the left column*]*, and subtract* [*the respective*] shi *for low and medium grades. The remainder is divided by the number of bundles of top grade cereal* [*in the right column*]*, yielding the* shi *for top grade cereal. The* shi *for all* [*grades*] *are each divided by the* fa *to yield the measures* [*per bundle of the respective grades*]*.*

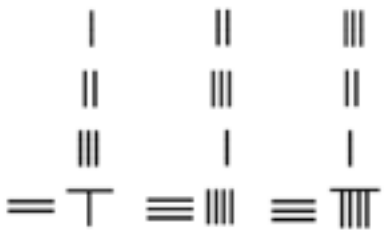
Whilst this may seem daunting at first, a careful consideration of the text gives us a full view of the familiar algorithm. To start, the answer states: '*Set up the columns in the centre and on the left in the same way as the column on the right*', requiring us to set up the 'matrix'. Instead of writing the equation from top left to bottom right, the ancient Chinese preferred to construct their text from top right to bottom left, and thus the original set of equations written in modern script

$$\begin{aligned} 3x + 2y + z &= 39; \\ 2x + 3y + z &= 34; \\ x + 2y + 3z &= 26, \end{aligned}$$

would become the following when written out in the ancient *fang cheng* form:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{array}$$

Of course, the ancient Chinese wouldn't have written it using the Hindu-Arabic numerals, not least because these had not yet been invented. Instead, they would have utilized the rod numeral system, and the above 'matrix' would be represented as:



where the numbers from $1$ to $9$ would be constructed as follows, with an empty space denoting zero:[2]

However, for the sake of clarity, I will continue the rest of this passage using the modern numerical system. Remarkably, this superficial difference in formatting the matrix forms the only departure from our usual weapon in dealing with this problem, as the solution provided by the book then reads:

> *Take the* [*number representing*] *top grade cereal in the right column to multiply all* [*numbers*] *in the central column, and then use* [*the method*] *of direct subtractions (*zhi chu*).*

This asks us to multiply all of the central column by $3$, the number representing the top-grade cereal in the right-hand column, and then using the method of *zhi chu*, usually translated as 'direct subtractions':

| | | |
|---|---|---|
| 1 | 6 | 3 |
| 2 | 9 | 2 |
| 3 | 3 | 1 |
| 26 | 102 | 39 |

Confusing translations of this concept aside, *zhi chu* simply refers to the act of repeated subtraction of one from the other until it is zero, and in this case, it is asking us to repeatedly subtract the right-hand column from the central column until the term at the top becomes

[2] Lam, 1994, 3–4.

zero. This equation only requires us to do this twice:

$$\begin{array}{ccc} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{array}$$

Thus, we have created a zero in the central column, and the exact same method is then subsequently applied to the left-hand column (*Once again multiply* [*the numbers*] *in the next column* [*that is, the left column, by the number representing top grade cereal in the right column*], *and then use* [*the method of*] *direct subtractions (*zhi chu*)*), giving us the following matrices:

$$\begin{array}{ccc} 3 & 0 & 3 \\ 6 & 5 & 2 \\ 9 & 1 & 1 \\ 78 & 24 & 39 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{array}$$

With two zeros on top of the left-hand and central columns, the book then asks us to create another zero in the second place in the left-hand column, and the way to achieve this is clear from the above: we simply multiply the left-hand column by $5$, the number representing medium grain in the central column, and repeatedly subtract the central column from the left-hand one until the zero appears in the second place (*Next multiply all* [*the numbers in*] *the left column by the remaining* [*number representing*] *medium grade cereal in the central column, and then use* [*the method of*] *direct subtractions*):

$$\begin{array}{ccc} 0 & 0 & 3 \\ 20 & 5 & 2 \\ 40 & 1 & 1 \\ 195 & 24 & 39 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{array}$$

From this point, it is obvious that we have arrived at the triangular form which signals the completion of our row reduction, and it is trivial then to find the solutions for each of the top, medium and low grade cereal, and the solution in the book then goes on to state:

> *To find* [*the measure for*] *medium grade cereal, multiply the* shi *in the central column by the* fa [*of the left column*], *and subtract the* shi *for low grade cereal. The remainder is divided by the number of bundles of medium grade cereal* [*in the central column*], *yielding the* shi *for medium grade cereal. To find* [*the measure for*] *top grade cereal, once again multiply the* shi *in the right column by the* fa [*of the left column*], *and subtract* [*the respective*] shi *for low and medium grades. The remainder is divided by the number of bundles of top grade cereal* [*in the right column*], *yielding the* shi *for top grade cereal. The* shi *for all* [*grades*] *are each divided by the* fa *to yield the measures* [*per bundle of the respective grades*].

The general method of the above can be easily summarised as finding the value for the low-grade cereal using the left-hand column, and then the medium grade using the central column, and the top grade the right-hand column. The passage given above is slightly different in terms of ordering to avoid introducing fractions in this particular question at too early a stage. However, this consideration is most likely only for the sake of convenience as opposed to some deep-seated fear of fractional values, since it was clear from the first chapter of the same book that the ancient Chinese were more than comfortable navigating the labyrinth of fractional divisions and multiplications.

With the full solution translated and explored, it is obvious that this method, for the lack of a better accolade that one might conjure, is just breathtakingly succinct, and the modern algorithm bears an uncanny resemblance to this solution. If we were simply to replace the example question from the Prelims Linear Algebra lecture notes with the above, it would be impossible to tell. Indeed, the only difference we can appreciate is that the solution in the lecture notes divided the subtrahend whereas the *Nine Chapters* multiplied the minuend, but that departure is as superficial as the difference in the matrix format.

The reader might, at this point, think that I have been overly enthusiastic about the subject, after all, just how impressive can solving simultaneous equations be? Surely something that we have learned in middle school cannot be held to such a high mathematical standard. However, even discounting the value that solving simultaneous equations could bring to the ancient Chinese society, the notation being used in carrying out these calculations was more than impressive enough. In many aspects, modern mathematics students are almost 'spoilt' when practicing the subject, much in the same way that we are by technological advances: we live in heated houses, air-conditioned rooms, have products from across the globe delivered to us, and access the collective vault of human knowledge with the internet. In terms of mathematics, we learn each theorem in an order that makes logical sense, as opposed to the date of their invention: we skip over centuries of academic turmoil and setbacks to reach the current 'version' of mathematics, and to top it all off, we have access to a system of notation that makes digesting all of this much easier than it would have been for our ancestors.

We take for granted the laying out of equations for the given problem and carrying out calculations using them. This was not the case when doing mathematics in various civilisations throughout history: what we would now express in numbers and letters was written in words and passages, bound by grammar and shackled by syntax, and solving problems under these conditions were much harder. The *Nine Chapters*' approach to simultaneous equations, however, suffered no such restraints. The *fang cheng* method, alongside its clear representation using counting rods, meant that the ancient Chinese were able to free themselves from the chains of semantics, and in many ways, this breakthrough allowed the early mathematicians to think in terms of symbols instead of words, which was a much clearer and more concise way to construct solutions.[3]

This one section in the *Nine Chapters* is far from the only noteworthy achievement in the book itself, and the *Nine Chapters* as a book was only a small part of the shining beacon that was ancient Chinese mathematics. At the risk of mimicking a nesting doll, I should also say that the mathematics practiced in this ancient kingdom was only a part of the marvellous work done throughout the antique world. Should the reader find themselves not yet sated by this

[3]Lam and Ang, 2004, 151–153.

article, then I recommend looking into the reading list below for those who would love to learn more about the history of mathematics.

### Recommended Reading List

Randy K. Schwartz, 'A Classic from China: The Nine Chapters', *Convergence*, December 2018, https://www.maa.org/press/periodicals/convergence/a-classic-from-china-the-nine-chapters

J. J. O'Connor and E. F. Robertson, '*Nine Chapters on the Mathematical Art*', December 2003, https://mathshistory.st-andrews.ac.uk/HistTopics/Nine_chapters

J. J. O'Connor and E. F. Robertson, 'Overview of Chinese mathematics', December 2003, https://mathshistory.st-andrews.ac.uk/HistTopics/Chinese_overview

David E. Joyce, 'Mathematics in China', September 1995, https://mathcs.clarku.edu/~djoyce/mathhist/china.html

### Bibliography

Lam, Lay Yong; Ang, Tian Se, *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, Revised Edition, World Scientific Publishing, 2004.

Lam, Lay Yong, 'Jiu Zhang Suanshu 九章算术 (*Nine Chapters on the Mathematical Art*): An Overview', *Archive for History of Exact Sciences*, vol. 47, no. 1, 1994, pp. 1–51. JSTOR: www.jstor.org/stable/41133972 (accessed 4 Aug 2020).

九章算术 (*Nine Chapters on the Mathematical Art*), as transcribed at https://ctext.org/nine-chapters (accessed 4 Aug 2020).