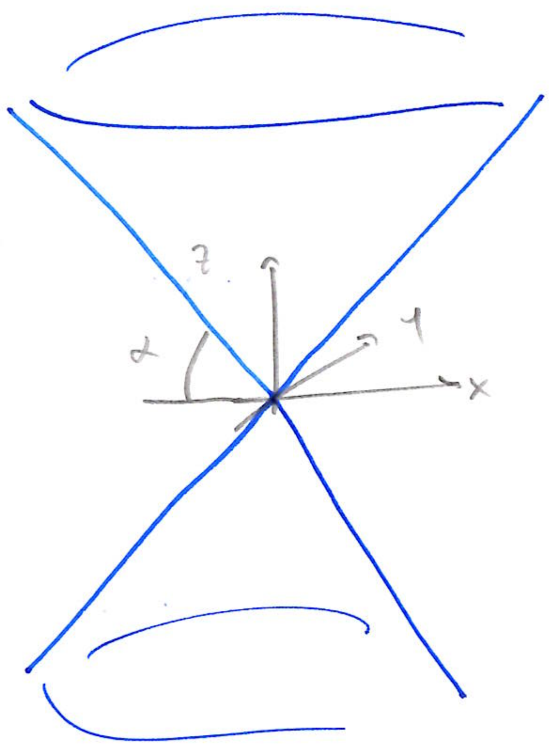
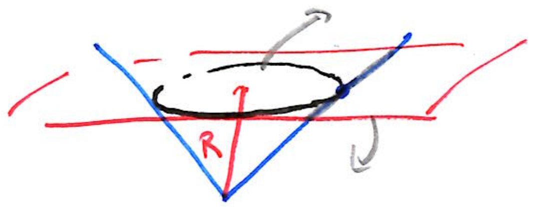
Conics (conic sections) - a family of planar curves

"Cone construction" - Formed by intersecting a plane w/ double cone

$$x^2 + y^2 = z^2 \cot^2\alpha$$



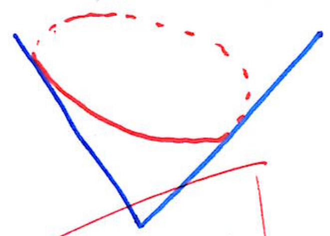
eg the plane $z = R$



→ circle

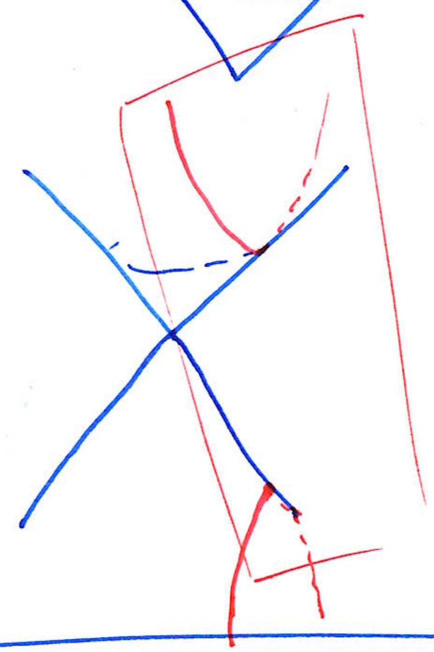
$x^2 + y^2 = R^2 \cot^2\alpha$

Now rotate plane: "small tilt"



→ ellipse

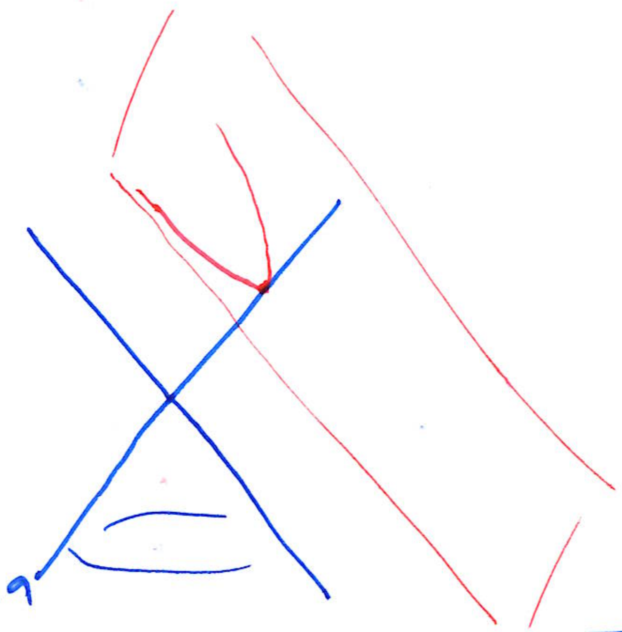
"large tilt"



→ 2 separated open curves . hyperbola

"A particular tilt"

→ a single open curve - parabola

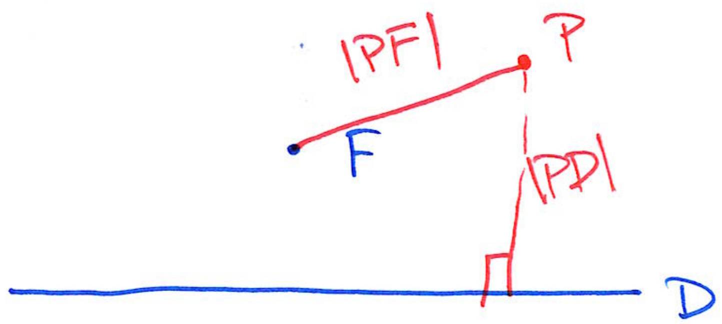


"Distance construction"

Ingredients : a line D (directrix)

a point F (focus)

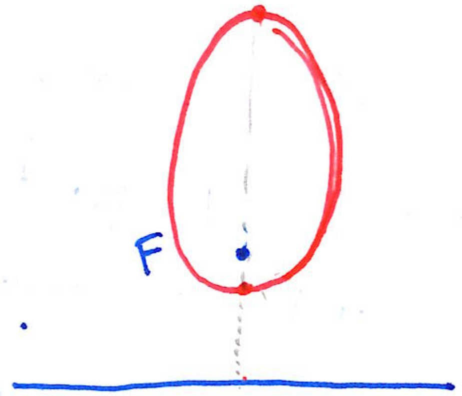
a number $e \geq 0$ (eccentricity)



Consider all pts P satisfying

$$|PF| = e\,|PD|$$

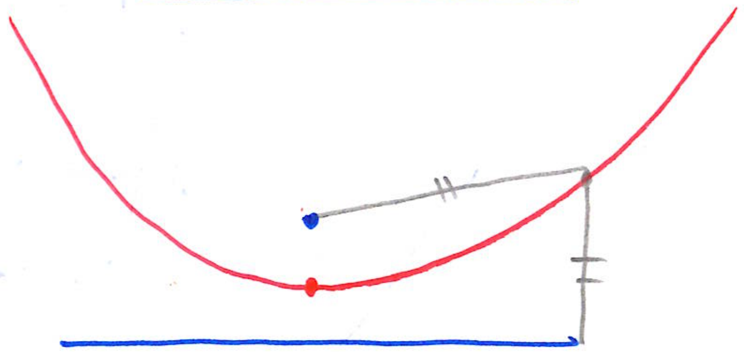
- $0 < e < 1$



As $e \searrow 0$, becomes more circular (& smaller)

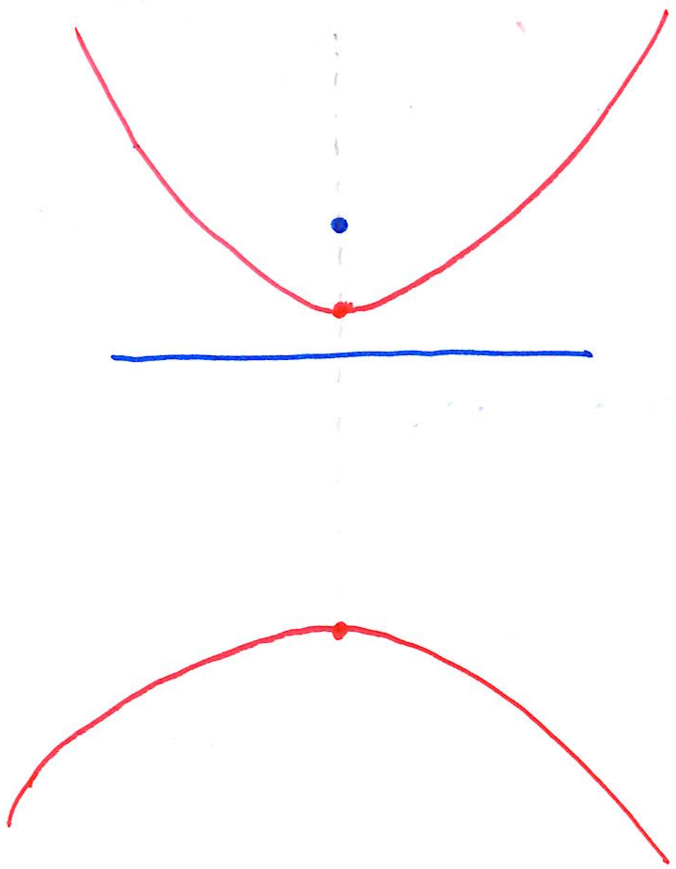
As $e \nearrow 1$, more & more elongated

- $e = 1$



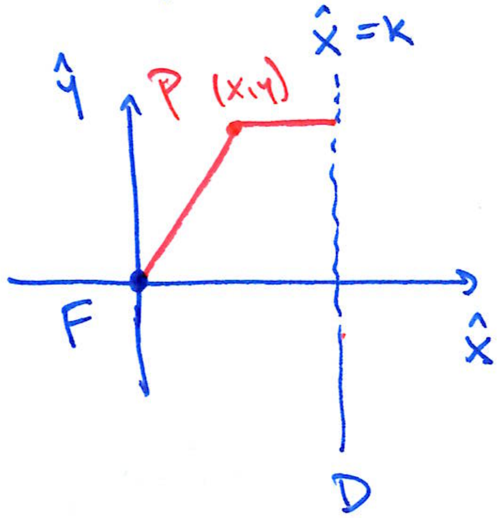
parabola

- $e > 1$



hyperbola

More formally ...



$|PF| = \sqrt{x^2 + y^2}$

$|PD| = k - x$ (if $x < k$)

So we want $\sqrt{x^2+y^2} = e(k-x)$

$\rightarrow x^2 + y^2 = e^2(k^2 - 2kx + x^2)$

$\rightarrow (1-e^2)x^2 + 2e^2kx + y^2 = e^2k^2$

$\rightarrow (1-e^2)\left(x + \dfrac{e^2k}{1-e^2}\right)^2 + y^2 = e^2k^2 + \dfrac{e^4k^2}{1-e^2} = \dfrac{e^2k^2}{1-e^2}$

compl. square

Now shift $x \mapsto x + \dfrac{e^2k}{1-e^2}$

$\rightarrow$ Normal form

$$(1-e^2)x^2 + y^2 = \frac{e^2k^2}{1-e^2}$$

$$\boxed{(1-e^2)x^2 + y^2 = \frac{e^2k^2}{1-e^2}} \quad (\star)$$

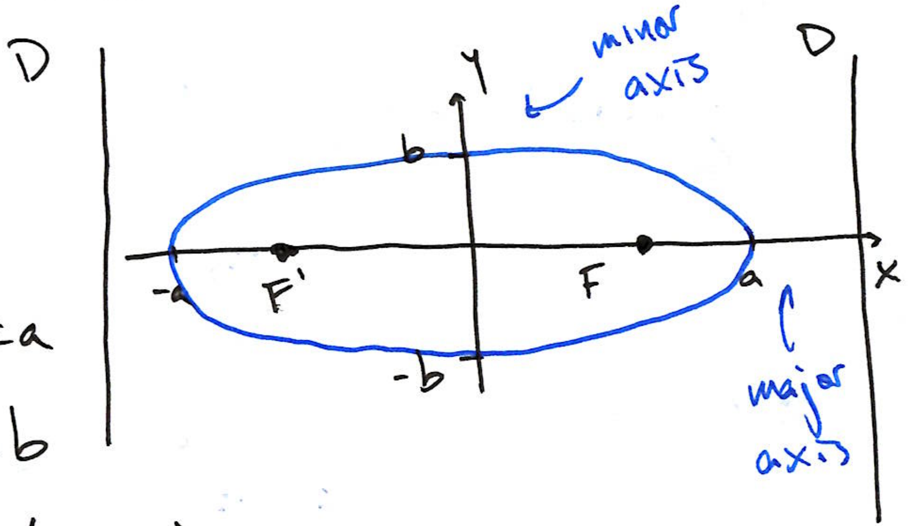
- **Ellipse** $0<e<1$

$\star$ can be written as

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

w/ $a = \dfrac{ke}{1-e^2}$, $b = \dfrac{ke}{\sqrt{1-e^2}}$

Notes:



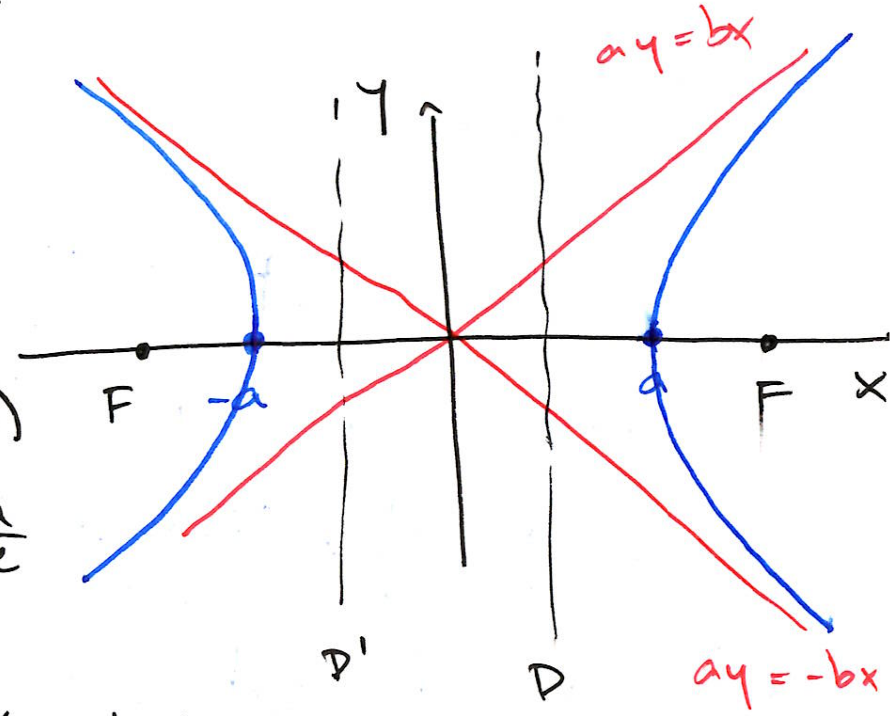
- $a > b$
- $y=0 \to x = \pm a$
- $x=0 \to y = \pm b$
- focus is at $(ae, 0)$
- $e = \sqrt{1 - \frac{b^2}{a^2}}$
- directrix : $x = \frac{k}{1-e^2} = \frac{a}{e}$
- Symmetry $\to$ another focus at $(-ae, 0)$
  " dir. at $x = -\frac{a}{e}$,
- Parameterisation : $x = a\cos t$, $y = b\sin t$, $t \in [0, 2\pi)$
- send $e \searrow 0$, $k \nearrow \infty$ st $ke = \ell$ held constant,
  a and b both approach ℓ, both foci collide at origin $\to$ circle rad. ℓ

- **Hyperbola** $(e>1)$ can write $\star$:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

, $a = \dfrac{ke}{e^2-1}$, $b = \dfrac{ke}{\sqrt{e^2-1}}$

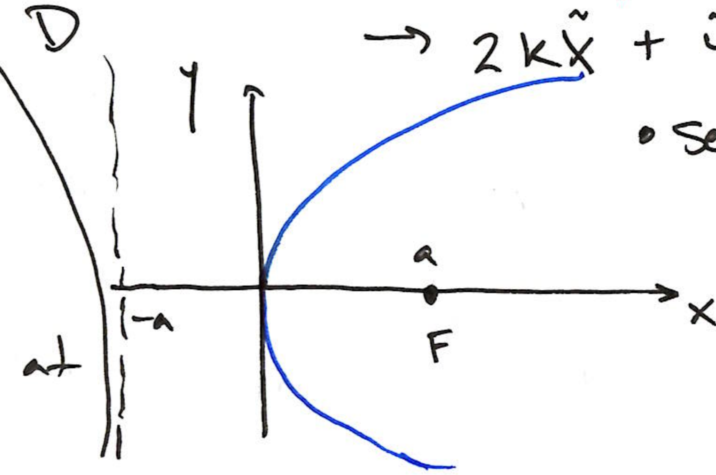
Notes



- $y=0 \to x = \pm a$
- $x=0 \to$ No sol'ns
- foci at $(\pm ae, 0)$
- directrix at $x = \pm \frac{a}{e}$
- ~~appro~~ approaches asymptotes at $\frac{x}{a} = \pm \frac{y}{b}$
- parameterisation : $x = a\cosh t$, $-\infty < t < \infty$
  $y = b\sinh t$

- **Parabola** $(e=1)$ Return to

$$\cancel{\tilde{x}^2} + \tilde{y}^2 = \underset{1}{e^2}(k^2 - 2k\tilde{x} + \cancel{\tilde{x}^2})$$

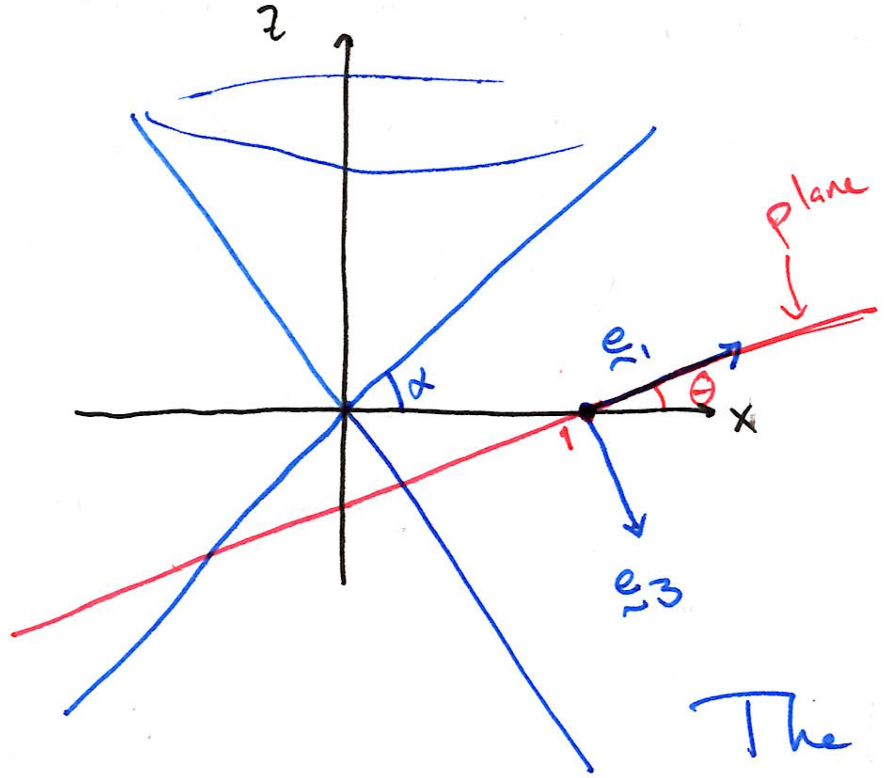


$\to 2k\tilde{x} + \tilde{y}^2 = k^2$

- set $a = \frac{k}{2}$, $x = a - \tilde{x}$
  $y = \tilde{y}$

$$\boxed{y^2 = 4ax}$$

Do the constructions agree? Let's verify the plane $z = \tan\theta \cdot (x-1)$ intersects the cone $x^2 + y^2 = z^2 \cot^2\alpha$ in an ellipse, parab., or hyperbola.



Let $\underset{\sim}{e}_1 = (\cos\theta, 0, \sin\theta)$, $\underset{\sim}{e}_2 = (0, 1, 0)$

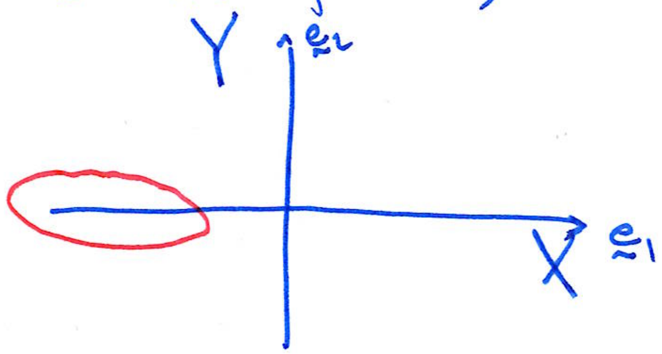
$\underset{\sim}{e}_3 = (\sin\theta, 0, -\cos\theta)$

- Observe: $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ are orthog. unit vec's in the plane ($\underset{\sim}{e}_3$ is perpen. to plane)

The point? · gives a coord. sys. for pts in plane

- if (x,y,z) is in plane, then $(x,y,z) = (1,0,0) + X\underset{\sim}{e}_1 + Y\underset{\sim}{e}_2 = (1 + X\cos\theta, Y, X\sin\theta)$

[in pic above,



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For pts also on cone, we have

$$(1 + X\cos\theta)^2 + Y^2 = X^2 \sin^2\theta \cot^2\alpha$$

. Now expand and complete square ...

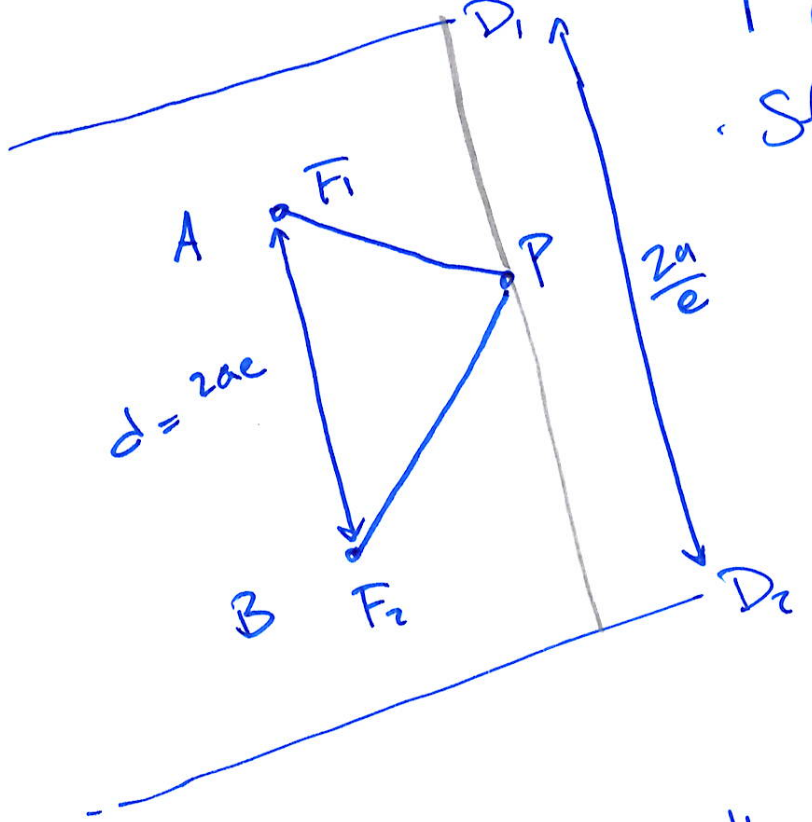
$$\underbrace{\frac{(\cos^2\theta - \sin^2\theta\cot^2\alpha)^2}{\sin^2\theta\cot^2\alpha}}_{=A}\left(X + \frac{\cos\theta}{\cos^2\theta - \sin^2\theta\cot^2\alpha}\right)^2 + \underbrace{\left(\frac{\cos^2\theta - \sin^2\theta\cot^2\alpha}{\sin^2\theta\cot^2\alpha}\right)}_{=B} Y^2 = 1$$

if $\theta \neq \alpha$

- $\theta < \alpha \Rightarrow A, B > 0$ $\rightarrow$ ellipse
- $\theta > \alpha \Rightarrow A > 0, B < 0$ $\Rightarrow$ hyperbola
- $\theta = \alpha \rightarrow 2X\cos\theta + Y^2 = -1$ $\rightsquigarrow$ parabola.

Ex 53. Let A, B be points in plane, a distance d apart.

Consider the locus of pts P satisfying $|PA| + |PB| = r > d$. Show that this is an ellipse w/ foci at A, B



$|AP| + |BP|$

$= e|PD_1| + e|PD_2|$

$= e|D_1D_2| = r$

– an ellipse w/ $a = \frac{r}{2}$

$e\frac{2a}{e} = r \Rightarrow a = \frac{r}{2}$

$d = 2ae \Rightarrow e = \frac{d}{2a} = \frac{d}{r}$

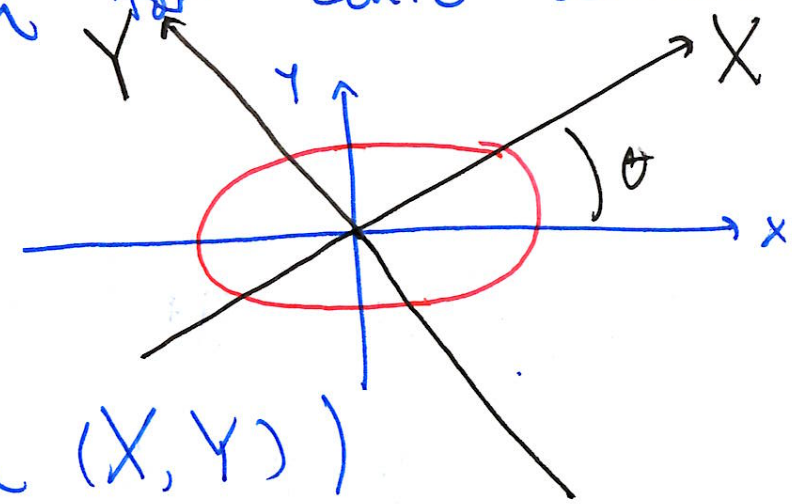
Degree Two Equation in Two Variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for A, B, C, D, E, F constants

Q. What set of points (eg curves) does it describe?

- includes conic sections
- for some constants, no sol'n, ie null set (eg $A=1$, $F=1$, all others 0)
- $B=0$ → can complete the square → normal form for conic section
- $B \neq 0$? Observe: - take "normal ellipse"
  - if rotate the axes, still an ellipse! (but won't be obvious in (X, Y))



→ idea: can rotate axes to remove B term?

rotation: $\begin{cases} X = x\cos\theta + y\sin\theta \\ Y = -x\sin\theta + y\cos\theta \end{cases}$

Goal: seek θ st $B=0$ in rotated coords

OR $\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases}$

· Let's consider reduced form: $Ax^2 + Bxy + Cy^2 = 1$ & suppose $A \geq C$ WLOG

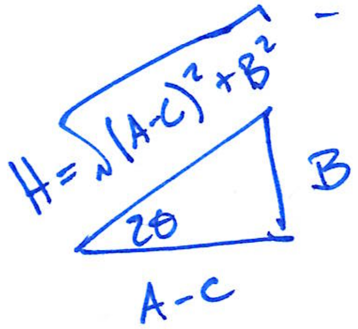
- in rotated variables: $\begin{bmatrix} c = \cos\theta \\ s = \sin\theta \end{bmatrix}$

$$(\star) \quad A(Xc - Ys)^2 + B(Xc - Ys)(Xs + Yc) + C(Xs + Yc)^2 = 1$$

Expand $\longrightarrow$ coeff. of XY term: $-2Acs + Bc^2 - Bs^2 + 2Ccs$

$$= B\cos 2\theta + (C - A)\sin 2\theta = 0 \quad \text{if} \quad \tan 2\theta = \frac{B}{A-C}$$

- can choose θ in range $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right]$ $\leftarrow \theta = \frac{\pi}{4}$ if $A = C$



$\Rightarrow \sin 2\theta = \frac{B}{H}$ $\quad$ $(\star)$ simplifies to $\left(\frac{A+C+H}{2}\right)X^2 + \left(\frac{A+C-H}{2}\right)Y^2 = 1$

$\cos 2\theta = \frac{A-C}{H}$

same sign if $(A+C+H)(A+C-H) > 0$

$\rightarrow (A+C)^2 > H^2 \rightarrow (A+C)^2 > (A-C)^2 + B^2$

$\rightarrow 4AC > B^2$

$\therefore$ We have cases:

(a) $B^2 - 4AC < 0 \rightarrow$ ellipse or empty set

(b) $B^2 - 4AC = 0 \rightarrow (A+C)^2 = H^2$ ie $A + C = H$ or $-H$

$\rightarrow$ one of coeff. $= 0 \longrightarrow \star$ simplifies to $\alpha X^2 = 1$ or $\beta Y^2 = 1$

which has solns: lines (eg $X = \pm\frac{1}{\alpha}$) or empty set

(c) $B^2 - 4AC > 0 \rightarrow$ different signs in front of X^2, Y^2, $\star$ has form $\alpha^2 X^2 - \beta^2 Y^2 = \pm 1$

always hyperbola

Ex 55 Find locus of $x^2 + xy + y^2 = 1$

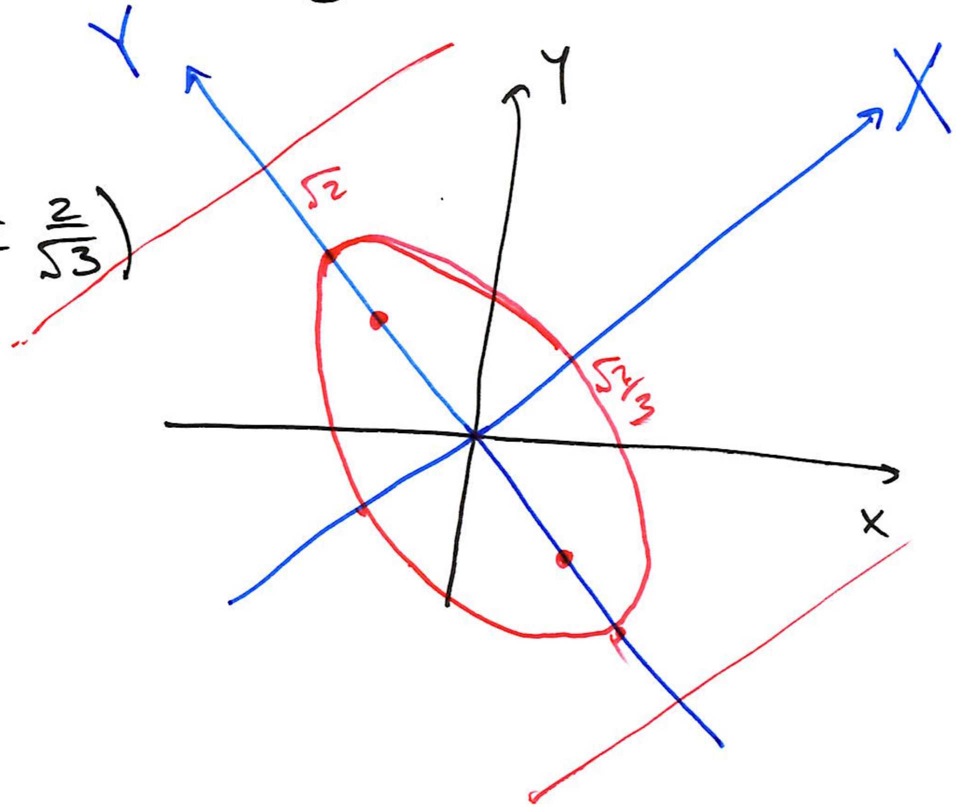
$A = 1 = B = C \rightarrow$ should rotate by $\theta = \frac{\pi}{4} \Rightarrow$ $x = \frac{\sqrt{2}}{2}(X - Y)$

$y = \frac{\sqrt{2}}{2}(X + Y)$

$H = \sqrt{(A-C)^2 + B^2} = 1 \quad \rightarrow \frac{3}{2}X^2 + \frac{1}{2}Y^2 = 1$

$$\rightarrow \Big(\underbrace{\frac{X}{\sqrt{\frac{2}{3}}}}_{= b}\Big)^2 + \Big(\underbrace{\frac{Y}{\sqrt{2}}}_{= a}\Big)^2 = 1$$

$\rightarrow$ ellipse w/ $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{2}{3}}$



foci : $(X, Y) = (0, \pm ae) = (0, \pm \frac{2}{\sqrt{3}})$

directrices : $Y = \pm \frac{a}{e} = \pm\sqrt{3}$