Surfaces

Objectives: construct, measure lengths & areas, extend idea of isometry

Surface descriptions / types:

- variable representation : $F(x,y,z)=0$

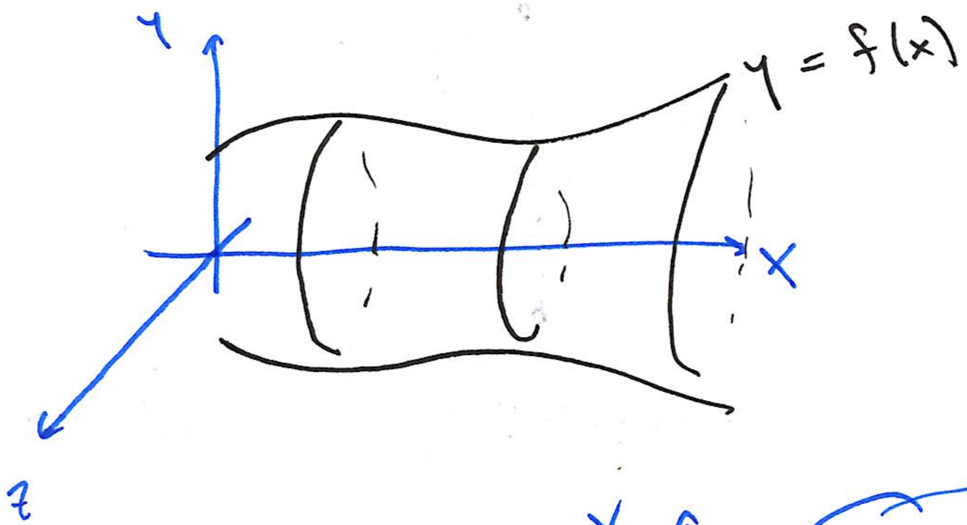
(more generally, $F(x_1, x_2, x_3) = 0$ ← could be other coords, eg polar)

eg $x^2+y^2+z^2=a^2$ (sphere)

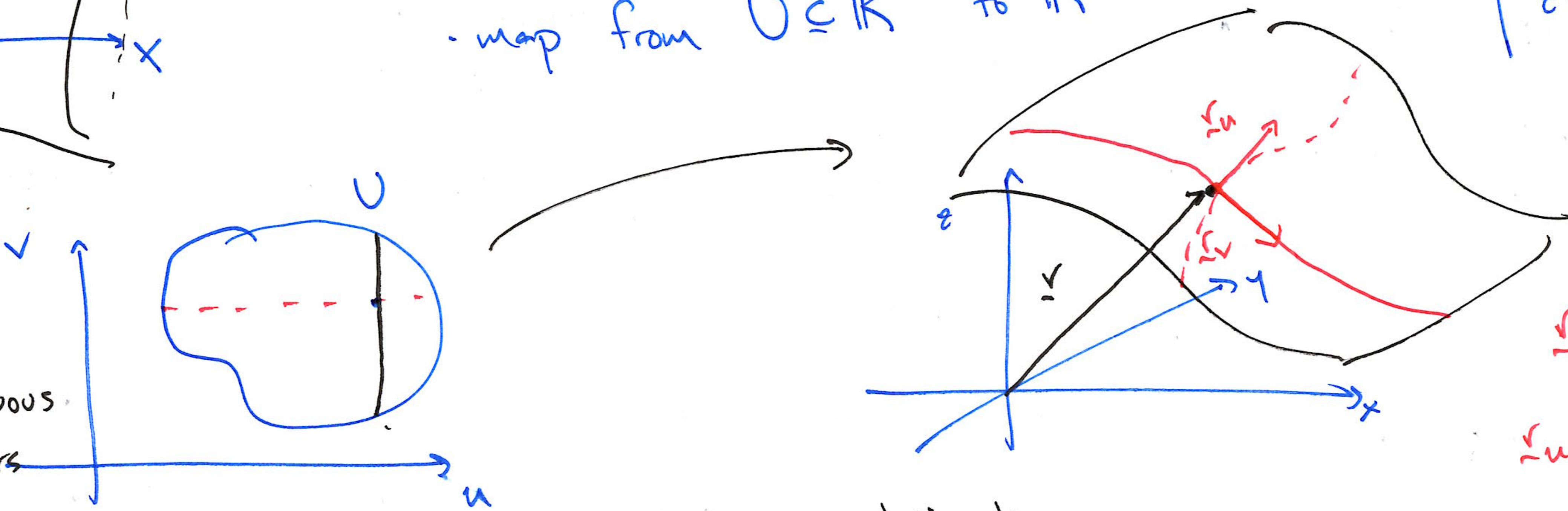
$z = x^2+y^2$ (paraboloid)

- graph $z = f(x,y)$

- Surface of revolution - rotate a curve about an axis



- Parametric representation $\underset{\sim}{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$
- map from $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$



tangent vectors

$\underset{\sim}{r}_v = \frac{\partial \underset{\sim}{r}}{\partial v}$

$\underset{\sim}{r}_u = \frac{\partial \underset{\sim}{r}}{\partial u}$

A smooth surface:

- x, y, z have continuous derivs of all orders
- $\underset{\sim}{r}_u$ and $\underset{\sim}{r}_v$ are lin. indep.

- map is a bijection

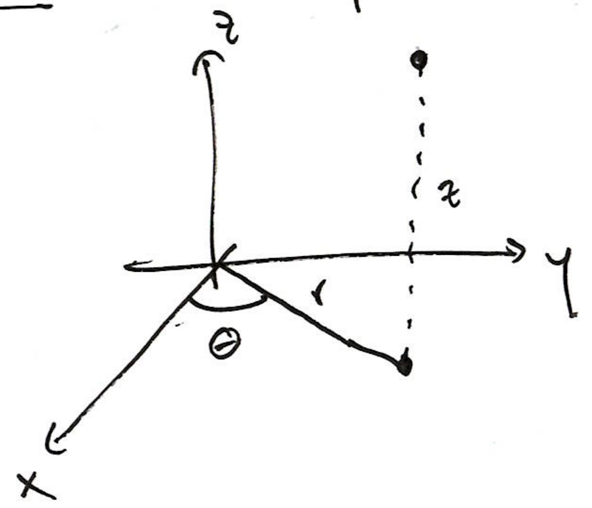
Notes:

- parameterisations <u>non</u>-unique
- a graph $z = f(x,y)$ can be parameterised by

$$\underset{\sim}{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

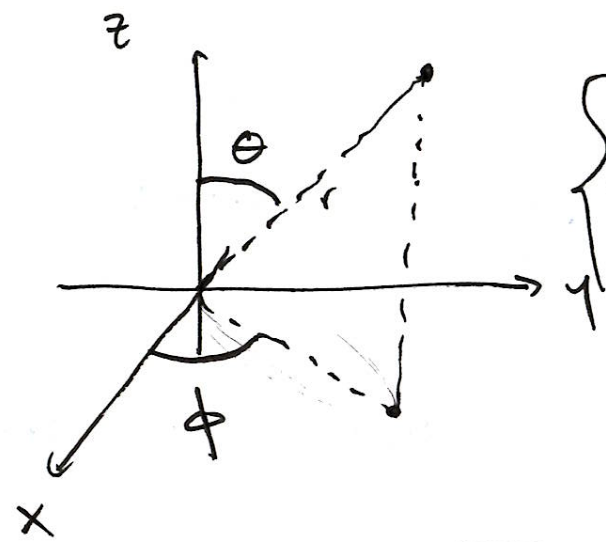
Recall

Cylindrical polar coords (r, θ, z)



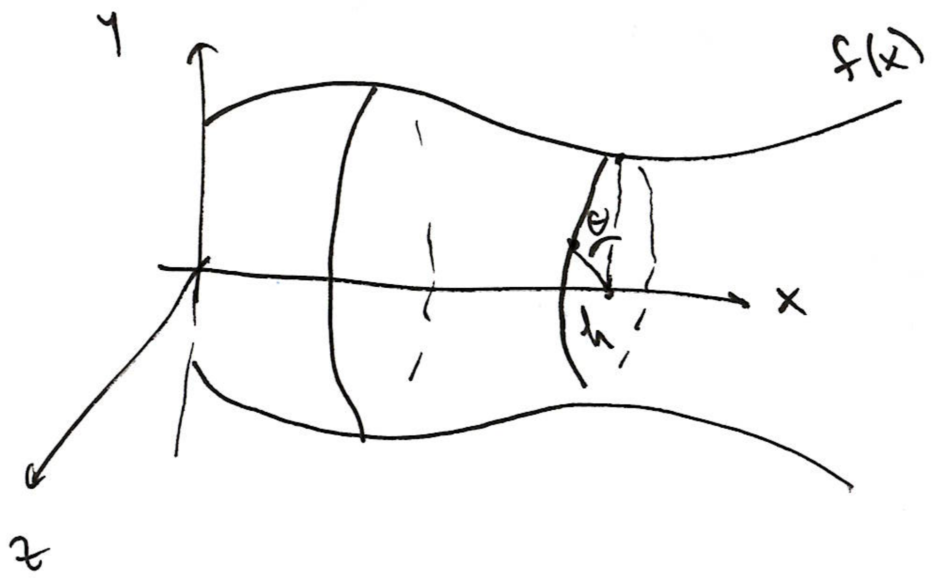
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Spherical coords (r, θ, ϕ)



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

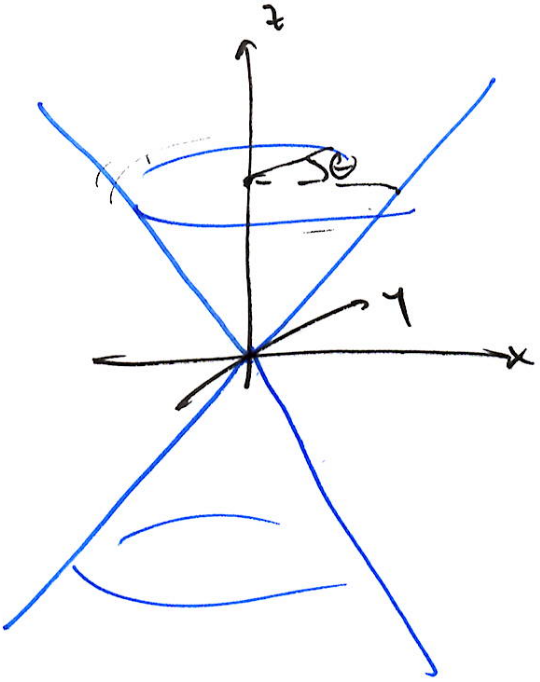
Consider surface of revolution - each cross section $x = h$ is a circle in plane parallel to y-z plane w/ radius $f(h)$



$\Rightarrow$ a natural parameterisation is

$$\underset{\sim}{r}(x,\theta) = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix}$$

Ex. Provide a parameterisation for the double cone $x^2+y^2=z^2$



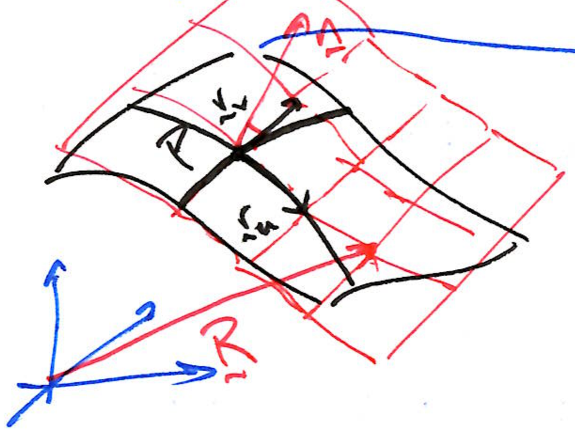
- Note - can't be expressed as a graph

$$\underset{\sim}{r}(\theta, z) = \begin{pmatrix} |z|\cos\theta \\ |z|\sin\theta \\ z \end{pmatrix}$$

$|z|$ implies a singularity point at origin

- not smooth at tip

Normal vector and tangent plane



- at any point on surface $\underline{r}(u,v)$, the two tangent vec's $\underline{r}_u$, $\underline{r}_v$ define a plane - the tangent plane

- then $\underline{r}_u \wedge \underline{r}_v$ forms a vector normal to the plane.

$\rightarrow$ we define the unit normal to surface : $\underline{n} = \dfrac{\underline{r}_u \wedge \underline{r}_v}{\|\underline{r}_u \wedge \underline{r}_v\|}$

OR $\underline{n} = \dfrac{\underline{r}_v \wedge \underline{r}_u}{\|\cdot\|}$

- Note: if surface has form $f(x,y,z)=0$, then $\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ is a normal vec to surface.

[Check: if $f = z - g(x,y)$, $\nabla f = \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix}$. Par: $\underline{r}(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix} \Rightarrow \underline{r}_x = \begin{pmatrix} 1 \\ 0 \\ g_x \end{pmatrix}$

$\underline{r}_y = \begin{pmatrix} 0 \\ 1 \\ g_y \end{pmatrix}$

And note $\underline{r}_x \cdot \nabla f = 0 = \underline{r}_y \cdot \nabla f$]

The plane tangent to point $\underline{P} = \underline{r}(\underline{p})$ has equation $(\underline{R} - \underline{P}) \cdot \left(\underline{r}_u(\underline{p}) - \underline{r}_v(\underline{p})\right) = 0$

OR $\underline{R}(a,b) = \underline{P} + a\,\underline{r}_u(\underline{p}) + b\,\underline{r}_v(\underline{p})$

Ex. Find tangent plane to ellipsoid $\dfrac{x^2}{3} + \dfrac{y^2}{2} + \dfrac{z^2}{6} = 1$ at point $(1,1,1)$

- $\nabla f = \left(\frac{2}{3}x, y, \frac{1}{3}z\right)$ is normal $\Rightarrow \nabla f(1,1,1) = \left(\frac{2}{3}, 1, \frac{1}{3}\right)$

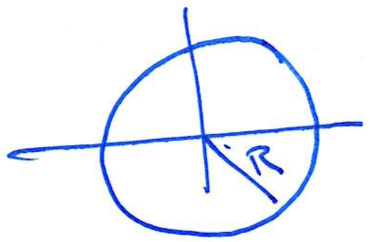
$\rightarrow$ plane : $\left((x,y,z) - (1,1,1)\right) \cdot \left(\frac{2}{3}, 1, \frac{1}{3}\right) = 0 \Rightarrow \underline{2x + 3y + z = 6}$

Arclength A curve $\underset{\sim}{\gamma}(t)$, $a \le t \le b$, has length $\int_a^b |\underset{\sim}{\gamma}'(t)|\, dt$. OR, starting from pt $t=a$, the arclength as a variable is $s(t) = \int_a^t |\underset{\sim}{\gamma}'(\tilde{t})|\, d\tilde{t}$

- $\underset{\sim}{\gamma}(t)$ is an arc-length parameterisation if $s(t) = t \quad \forall t$

  true iff $|\underset{\sim}{\gamma}'(t)| \equiv 1$

Compare:



$$\underset{\sim}{\gamma}_1(t) = R\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \quad 0 \le t < 2\pi$$

$$\underset{\sim}{\gamma}_2(s) = R\begin{pmatrix}\cos\left(\frac{s}{R}\right)\\ \sin\left(\frac{s}{R}\right)\end{pmatrix}, \quad 0 \le s < 2\pi R$$

Note $|\underset{\sim}{\gamma}_2'(s)| \equiv 1$ - arclength par.

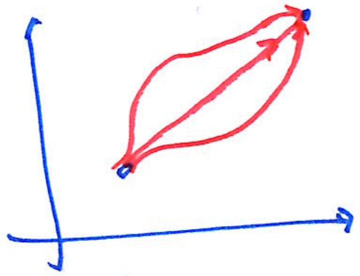
- Converting to arc length par: $\frac{ds}{dt} = |\underset{\sim}{\gamma}'(t)|$, $s(0)=0$ $\xrightarrow{\text{solve}} s(t) \xrightarrow{\text{invert}} t(s) \rightarrow \hat{\underset{\sim}{\gamma}}(s) = \underset{\sim}{\gamma}(t(s))$ is arclength par.

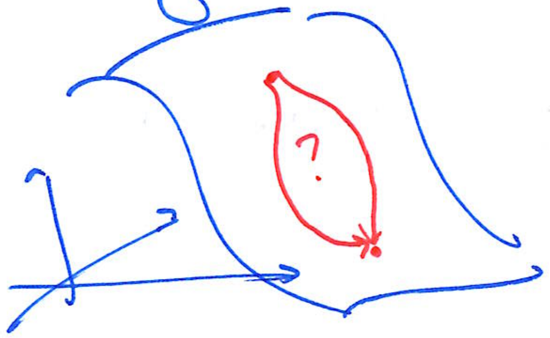
[in previous ex: $|\underset{\sim}{\gamma}_1'(t)| = R = \frac{ds}{dt} \Rightarrow s = Rt \Rightarrow t = \frac{s}{R}$

$$\rightarrow \underset{\sim}{\gamma}_2(s) = \underset{\sim}{\gamma}_1\left(\frac{s}{R}\right)$$

Shortest Path Between 2 Points on a Surface

On a plane - straight line! (Can you prove it? - Part A Calc of Variations)





On curved surface?

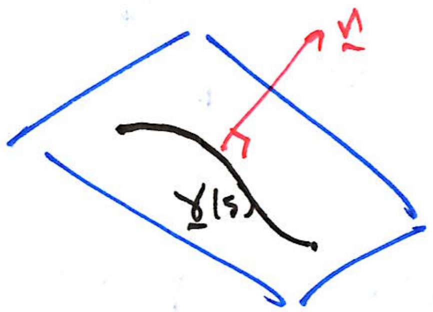
- called geodesic

Thm Let $\underset{\sim}{\gamma}(s)$ be a curve on surface w/ arc length parameterisation,

• $\underset{\sim}{\gamma}(s)$ is a geodesic iff $\underset{\sim}{\gamma}''(s) \wedge \underset{\sim}{n}(s) = \underset{\sim}{0}$ where $\underset{\sim}{n}(s)$ is normal vec.

On a plane: $\underset{\sim}{\gamma}''(s)$ is in the plane, thus $\underset{\sim}{\gamma}'' \wedge \underset{\sim}{n} = \underset{\sim}{0}$ iff $\underset{\sim}{\gamma}'' = \underset{\sim}{0}$ (ie $\underset{\sim}{\gamma}' = \underset{\sim}{\text{const}}$)

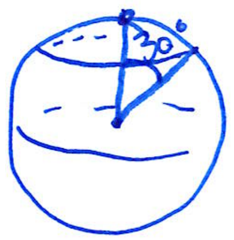
iff $\underset{\sim}{\gamma}$ is a straight line



Ex 99 Compare 2 paths from $60°N, 0°E$ (in North Sea) to $60°N, 90°E$ (in Russia)

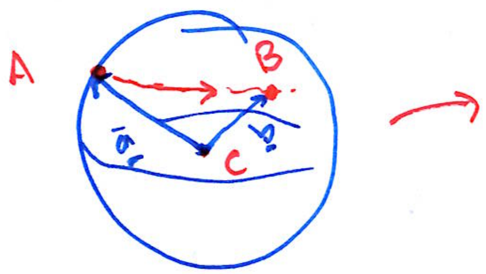
Path 1: follow 60°N Path 2: Great circle

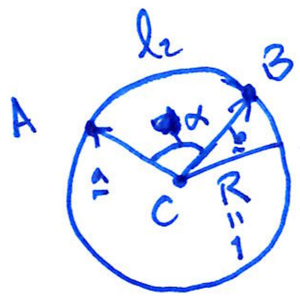
→ in spherical coords: $(R = 1 \text{ WLOG})$ $\theta = 30° = \frac{\pi}{6}$ rad, $\phi \in [0, \frac{\pi}{2}]$



$$\underset{\sim}{\gamma}_1(\phi) = \left(\tfrac{1}{2}\cos\phi, \tfrac{1}{2}\sin\phi, \tfrac{\sqrt{3}}{2}\right) \Rightarrow l_1 = \int_0^{\pi/2} |\underset{\sim}{\gamma}_1'(\phi)|\, d\phi = \frac{\pi}{4} \approx \boxed{.785}$$

Great Circle - circle passing thru 2 points and contains centre of sphere





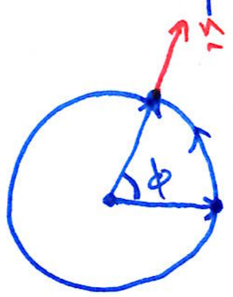
$l_2 = R\alpha = \alpha$

$\underset{\sim}{a} = (\tfrac{1}{2}, 0, \tfrac{\sqrt{3}}{2})$ $(\phi = 0 \text{ in } \underset{\sim}{\gamma}_1)$

$\underset{\sim}{b} = (0, \tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$ $(\phi = \frac{\pi}{2})$

$$\Rightarrow \alpha = \cos^{-1}(\underset{\sim}{a} \cdot \underset{\sim}{b}) = \cos^{-1}\left(\frac{3}{4}\right) \approx \boxed{.72 = l_2}$$

- Note: A path on any great circle could be written

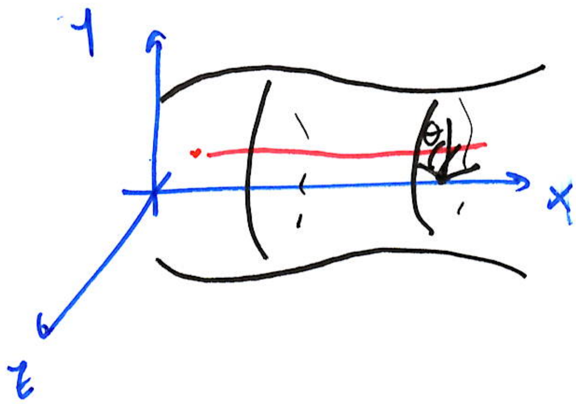


$\underset{\sim}{\gamma}(\phi) = (\cos\phi, \sin\phi, 0)$ → $\underset{\sim}{\gamma}''(\phi) = -\underset{\sim}{\gamma}(\phi) = -\underset{\sim}{n}$

~~arcle~~ arc length par.

$\therefore \underset{\sim}{\gamma}'' \wedge \underset{\sim}{n} = \underset{\sim}{0}$ ∴ Great circle is a geodesic

Ex. Show that on a surface of revolution, meridians (curves w/ fixed angle) are geodesics.



- if rotate $y = F(x)$ about x-axis, fix angle $\theta = \alpha$,

$$\to \text{curve} \quad \underset{\sim}{\gamma}(x) = \begin{pmatrix} x \\ F(x)\cos\alpha \\ F(x)\sin\alpha \end{pmatrix}$$

- But not an arclength par!

To obtain arclength par: parameterise planar curve as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}$ st $f'^2 + g'^2 \equiv 1$

$$\to \text{surface} : \underset{\sim}{r}(s,\theta) = \begin{pmatrix} f(s) \\ g(s)\cos\theta \\ g(s)\sin\theta \end{pmatrix}$$

A meridian: $\underset{\sim}{\gamma}(s) = \underset{\sim}{r}(s,\alpha)$ for fixed α

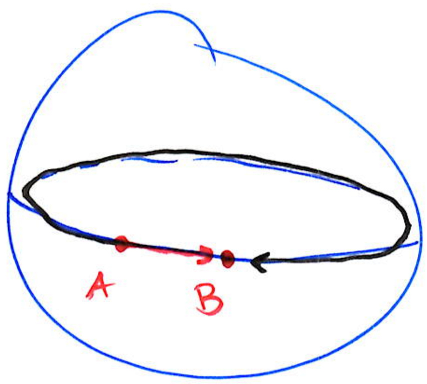
Now can compute $\underset{\sim}{\gamma}''$, $\underset{\sim}{n} = \underset{\sim}{r}_s \wedge \underset{\sim}{r}_\theta$ w/ $\theta = \alpha$

Then find $\underset{\sim}{\gamma}'' \wedge \underset{\sim}{n} = \underset{\sim}{0}$ $\forall s$.

[uses : $f'^2 + g'^2 = 1 \Rightarrow 2f'f'' + 2g'g'' = 0$]

<u>Geodesics</u> - a curve $\underset{\sim}{\gamma}(t)$ for which $\underset{\sim}{\gamma}''(t) \wedge \underset{\sim}{n}(t) = \underset{\sim}{0}$ at each point.

- The shortest path between 2 points on a surface will be a geodesic. (But geodesic does not imply shortest path)



Geodesic A path $\underset{\sim}{\gamma}(t)$ for which $\underset{\sim}{\gamma}'' \wedge \underset{\sim}{n} = \underset{\sim}{0}$

Claim $|\underset{\sim}{\gamma}'(t)| = $ constant

Pf $\frac{d}{dt}|\underset{\sim}{\gamma}'(t)|^2 = \frac{d}{dt}(\underset{\sim}{\gamma}' \cdot \underset{\sim}{\gamma}') = 2\underset{\sim}{\gamma}'' \cdot \underset{\sim}{\gamma}' = 0$

Recall : $|\underset{\sim}{\gamma}'(s)| = 1$ for arc length par.

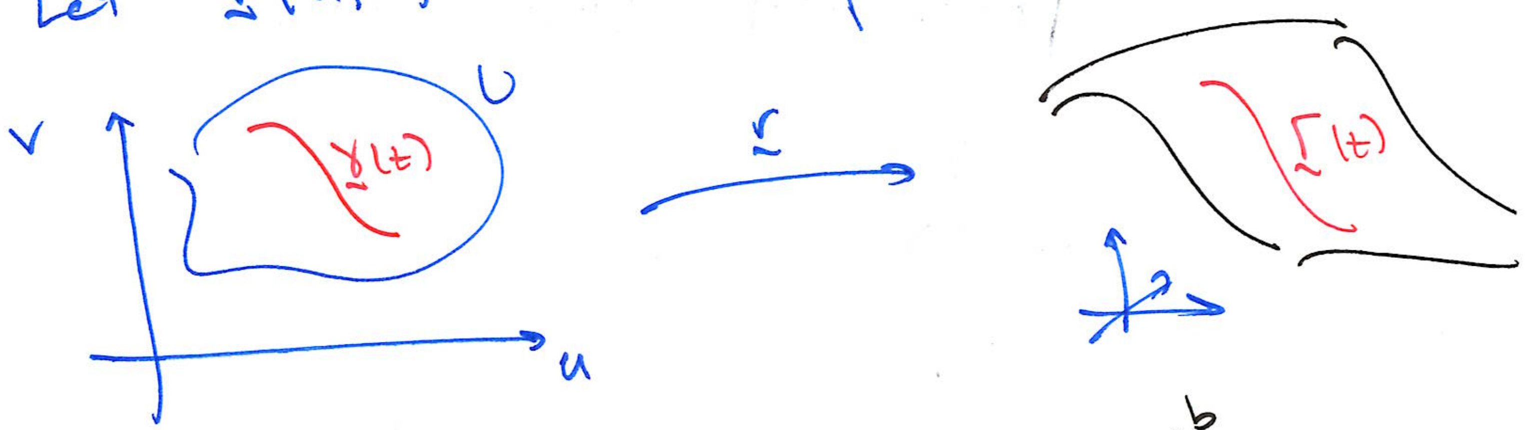
$\therefore$ if $\underset{\sim}{\gamma}(t)$ is a geodesic w/ $|\underset{\sim}{\gamma}'(t)| = c \neq 1$,

then $s = ct \Rightarrow \underset{\sim}{\hat{\gamma}}(s) = \underset{\sim}{\gamma}\left(\frac{s}{c}\right)$ will be

arc length par & geodesic

Surface Isometry - A map that transforms one surface into another is an isometry if it preserves lengths.

Let $\underset{\sim}{r}(u,v)$ be a par. of a surface



$$\underset{\sim}{\gamma}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

$$\downarrow$$

$$\underset{\sim}{\Gamma}(t) = \underset{\sim}{r}(u(t), v(t))$$

$$l = \int_a^b |\underset{\sim}{\gamma}'(t)|\, dt \qquad L = \int_a^b |\underset{\sim}{\Gamma}'(t)|\, dt$$

isometry if $l = L$ for all paths $\underset{\sim}{\gamma}$ iff $|\underset{\sim}{\gamma}'(t)| = |\underset{\sim}{\Gamma}'(t)| \;\forall \underset{\sim}{\gamma}$

$$|\underset{\sim}{\gamma}'(t)|^2 = u'(t)^2 + v'(t)^2$$

$$|\underset{\sim}{\Gamma}'(t)|^2 = |\underset{\sim}{r}_u u'(t) + \underset{\sim}{r}_v v'(t)|^2 = \ldots \overset{?}{=} u'(t)^2 + v'(t)^2$$

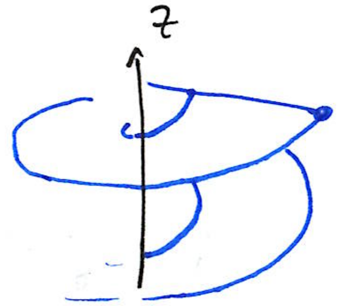
if yes, then isometry

if no, not isometry

Ex 101

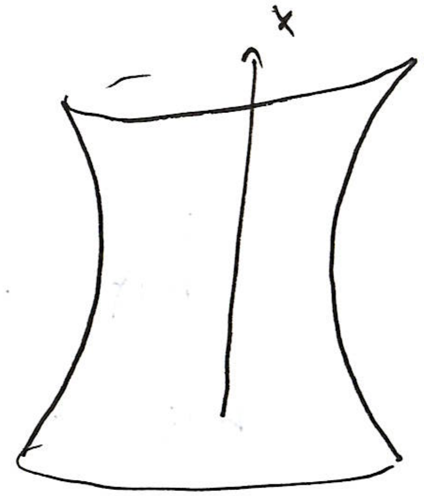
Helicoid

$$\underline{s}(X,Z) = \begin{pmatrix} X\cos Z \\ X \sin Z \\ Z \end{pmatrix}$$

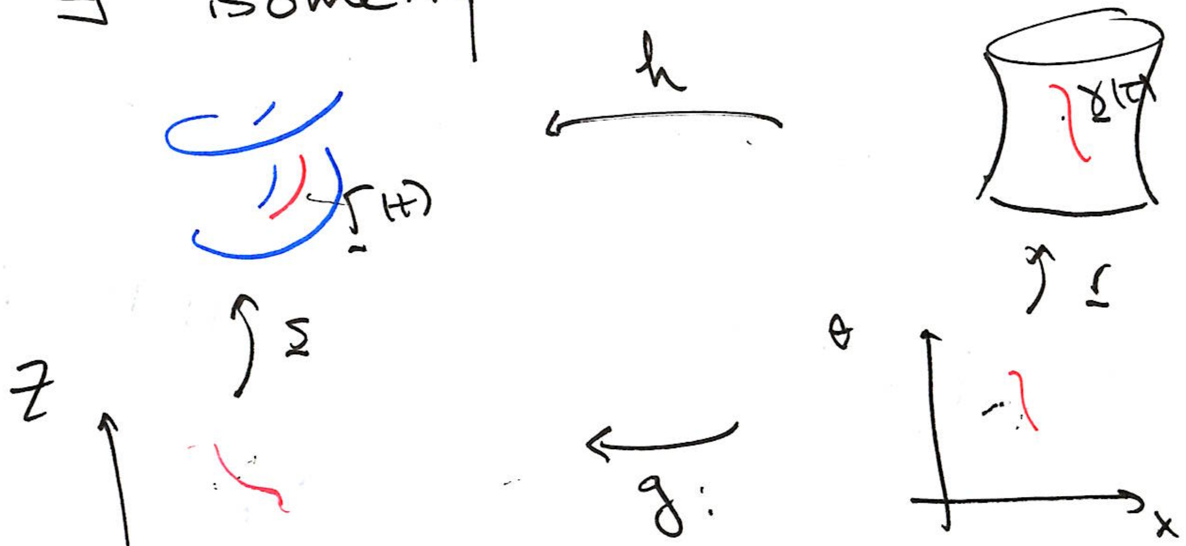


Catenoid

$$\underline{r}(x,\theta) = \begin{pmatrix} x \\ \cosh x \cos\theta \\ \cosh x \sin\theta \end{pmatrix}$$



Claim $\exists$ isometry between these surfaces.



$X(x,\theta)$

$Z(x,\theta)$

$$h = s \circ g \circ r^{-1}$$

h is an isometry if lengths are preserved: $|\underline{\gamma}'(t)| = |\underline{\Gamma}'(t)|$

$$\underline{\gamma}(t) = \underline{r}(x(t), \theta(t))$$

$$\Rightarrow \underline{\gamma}'(t) = \underline{r}_x x'(t) + \underline{r}_\theta \theta'(t)$$

$$\underline{\Gamma}(t) = \underline{s}\big(X(x(t),\theta(t)), Z(x(t),\theta(t))\big)$$

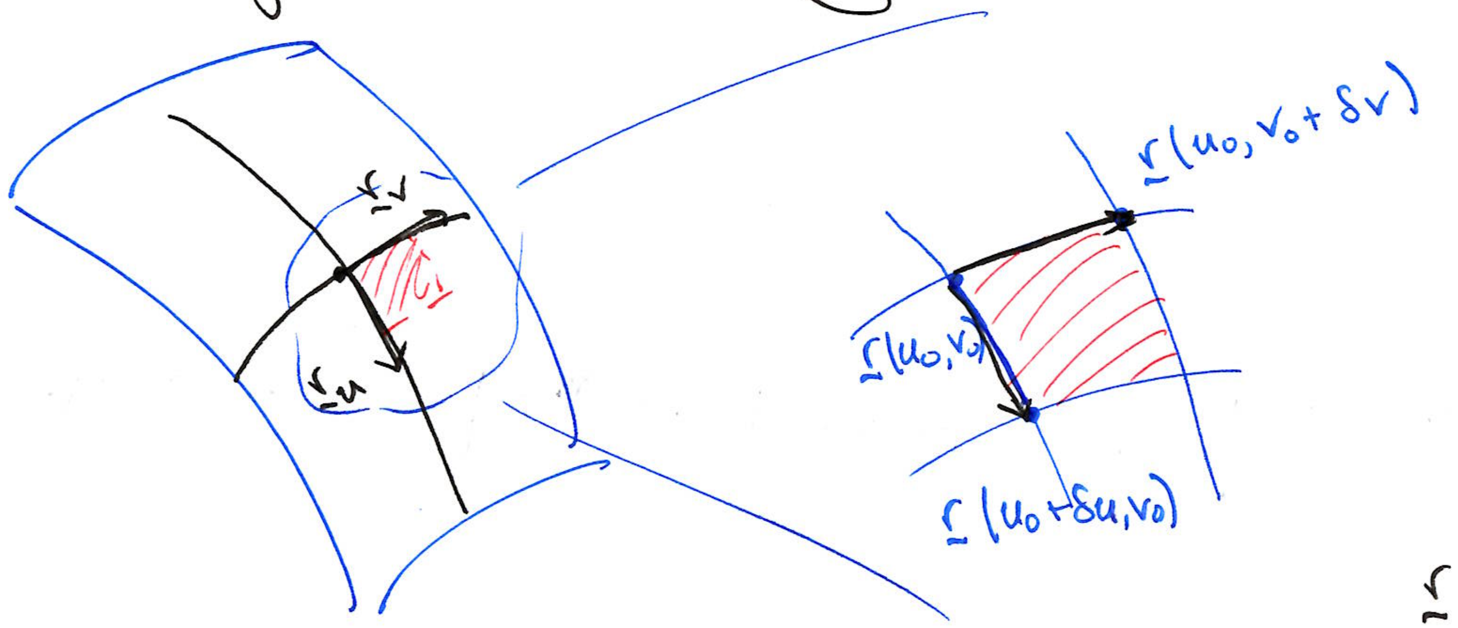
$$\Rightarrow \underline{\Gamma}'(t) = \underline{s}_X\left(\frac{\partial X}{\partial x}x'(t) + \frac{\partial X}{\partial \theta}\theta'(t)\right) + \underline{s}_Z\left(\frac{\partial Z}{\partial x}x'(t) + \frac{\partial Z}{\partial \theta}\theta'(t)\right)$$

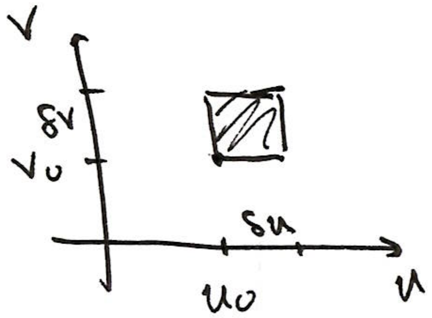
- Isometry for g: $\begin{cases} X = \sinh x \\ Z = \theta \end{cases}$

Surface Area

Surface $\underset{\sim}{r}(u,v)$, $u,v \in U \subseteq \mathbb{R}^2$

Imagine zooming in on a patch of surface corresponding to region $u \in [u_0, u_0+\delta u]$, $v \in [v_0, v_0+\delta v]$





Expanding:

$$\underset{\sim}{r}(u_0+\delta u, v_0) - \underset{\sim}{r}(u_0,v_0) \approx \frac{\partial \underset{\sim}{r}}{\partial u}(u_0,v_0)\,\delta u$$

$$\underset{\sim}{r}(u_0, v_0+\delta v) - \underset{\sim}{r}(u_0,v_0) \approx \frac{\partial \underset{\sim}{r}}{\partial v}(u_0,v_0)\,\delta v$$

The patch is well-approximated by parallelogram w/ sides $\underset{\sim}{r}_u\,\delta u$, $\underset{\sim}{r}_v\,\delta v$ $\Rightarrow$ area $\approx |\underset{\sim}{r}_u\,\delta u \wedge \underset{\sim}{r}_v\,\delta v|$

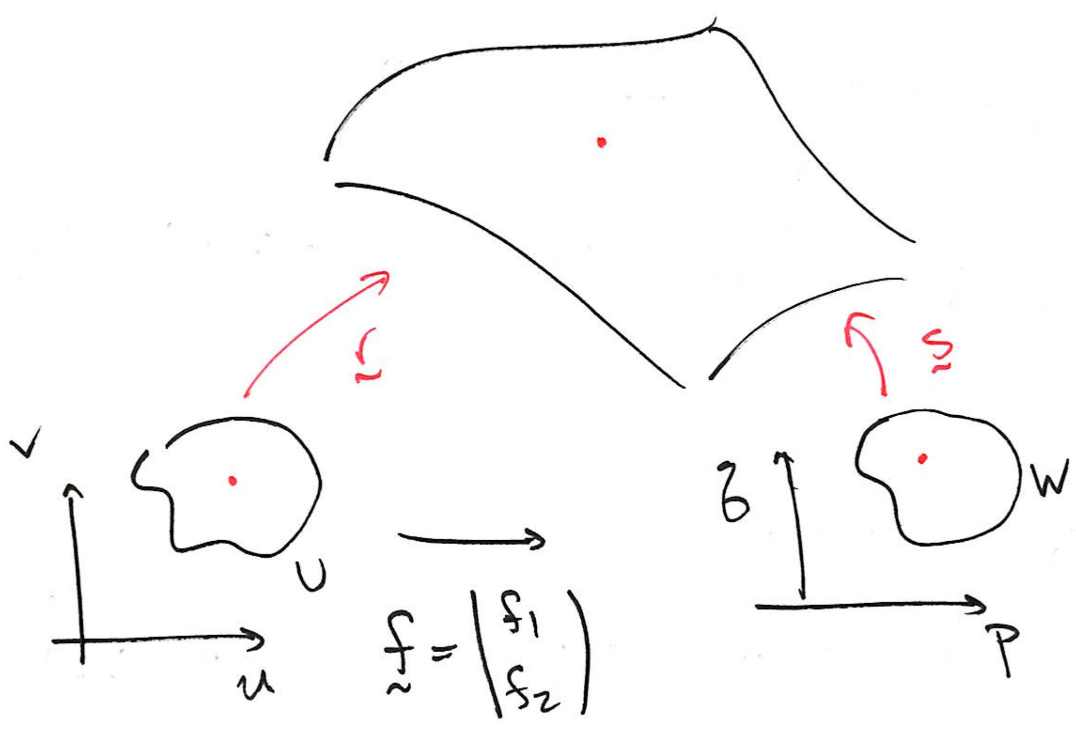
$$= |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v|\,\delta u\,\delta v$$

The surface area of $\underset{\sim}{r}(U)$ for $U \subseteq \mathbb{R}^2$ is

$$\boxed{A = \iint_U |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v|\,du\,dv}$$

Notation: $dS = |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v|\,du\,dv$ is infinitesimal surface area element

★ Surface area is independent of parameterisation



So $\underset{\sim}{r}(u,v) = \underset{\sim}{s}(\underset{\sim}{f}(u,v))$

$= \underset{\sim}{s}(\underbrace{f_1(u,v)}_{p}, \underbrace{f_2(u,v)}_{z})$

Chain rule: $\underset{\sim}{r}_u = \underset{\sim}{s}_p f_{1u} + \underset{\sim}{s}_z f_{2u}$

$$\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v = \frac{\partial(p,z)}{\partial(u,v)} \underset{\sim}{s}_p \wedge \underset{\sim}{s}_z \qquad \begin{vmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = p_u z_v - p_v z_u$$

$$\therefore \text{Area} = \iint_U |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v| \, du\,dv = \iint_U |\underset{\sim}{s}_p \wedge \underset{\sim}{s}_z| \underbrace{\left|\frac{\partial(p,z)}{\partial(u,v)}\right| du\,dv}_{dp\,dz} = \iint_W |\underset{\sim}{s}_p \wedge \underset{\sim}{s}_z| \, dp\,dz$$

$$\text{Surface Area} = \iint_U |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v| \, du\, dv$$

1. Graph $z = f(x,y)$, $x, y \in S$

$$\underset{\sim}{r}(x,y) = (x, y, f(x,y)) \rightarrow A = \iint_S |\underset{\sim}{r}_x \wedge \underset{\sim}{r}_y| \, dx\,dy = \iint_S \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy$$

2. Surface of Revolution. rotate $y = f(x)$ about x-axis

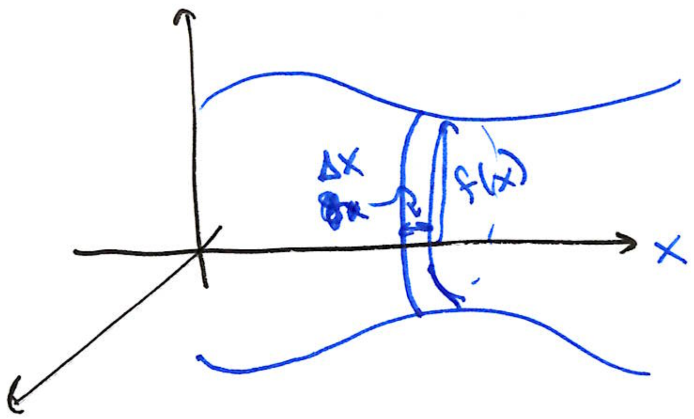
$$\underset{\sim}{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta) \rightarrow |\underset{\sim}{r}_x \wedge \underset{\sim}{r}_\theta| = f(x)\sqrt{1 + f'(x)^2}$$

$$\Rightarrow A = 2\pi \int_a^b f(x)\sqrt{1 + f'^2} \, dx$$

[Note: arc length of meridian

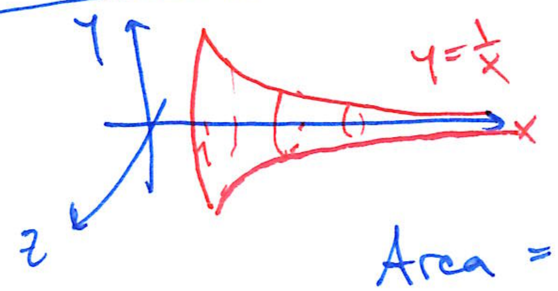
$$s(x) = \int_a^x \sqrt{1 + f'^2} \, d\tilde{x} \rightarrow \frac{ds}{dx} = \sqrt{1 + f'^2}$$

$$\rightarrow A = 2\pi \int_a^b f(x) \frac{ds}{dx} \, dx$$



Area is NOT obtained by approximating area of disc as $2\pi f(x)\Delta x$!

Gabriel's Horn rotate $y = \frac{1}{x}$, $x \geq 1$ about x-axis.

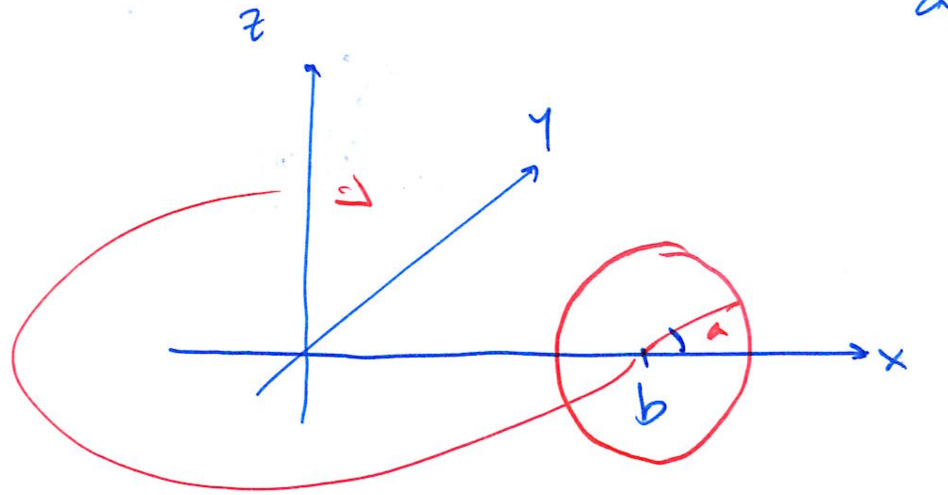


$$\text{Volume} = \pi \int_1^\infty \frac{1}{x^2} \, dx = \pi$$

$$\text{Area} = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \, dx > 2\pi \int_1^\infty \frac{1}{x} \, dx \rightarrow \infty$$

Compare: $\text{volume} = \pi \int_a^b f(x)^2 \, dx$

Ex. Area of torus : rotate circle $(x-b)^2 + z^2 = a^2$ $(a<b)$ about z-axis



parameterisation : $\underset{\sim}{r}(\theta,\phi) = \begin{pmatrix} (b+a\cos\theta)\cos\phi \\ (b+a\cos\theta)\sin\phi \\ a\sin\theta \end{pmatrix}$

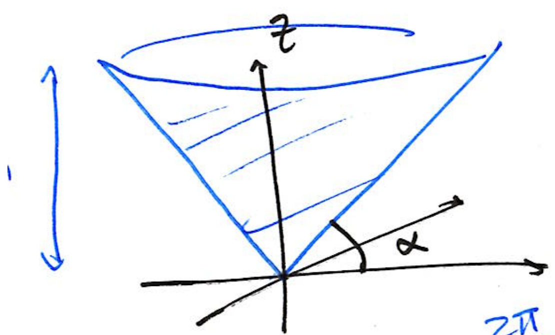
$0<\theta,\phi<2\pi$

(PS 6)

$|\underset{\sim}{r}_\theta \wedge \underset{\sim}{r}_\phi| = a(b+a\cos\theta)$

$\Rightarrow A = 2\pi a \int_0^{2\pi} (b+a\cos\theta)\,d\theta = \underline{\underline{4\pi^2 ab}} + 2\pi a^2 \cancel{\int_0^{2\pi}\cos\theta\,d\theta}^{\,0}$

Ex. Area of cone $x^2+y^2 = z^2\cot^2\alpha$, $0 \le z \le h$

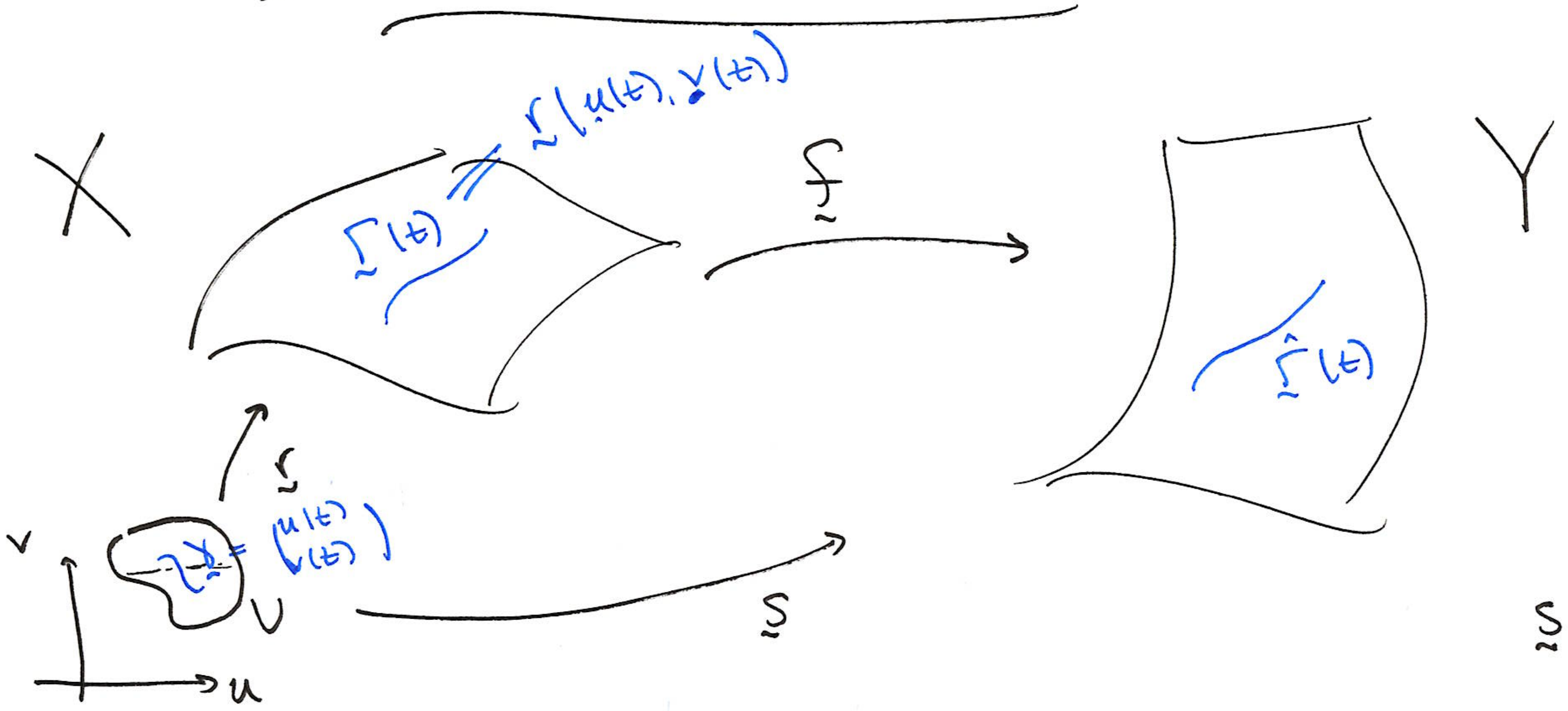


par : $\underset{\sim}{r}(z,\theta) = \begin{pmatrix} z\cot\alpha\cos\theta \\ z\cot\alpha\sin\theta \\ z \end{pmatrix}$ $\rightarrow$ $\underset{\sim}{r}_z \wedge \underset{\sim}{r}_\theta = \begin{pmatrix} -z\cot\alpha\cos\theta \\ -z\cot\alpha\sin\theta \\ z\cot^2\alpha \end{pmatrix}$

$\rightarrow A = \int_0^{2\pi}\int_0^{h} z\cot\alpha\sqrt{1+\cot^2\alpha}\,dz\,d\theta = \dfrac{\pi h^2\cos\alpha}{\sin^2\alpha}$

(Note: $A \nearrow \infty$ as $\alpha \downarrow 0$)

Isometries and Area



Claim: Isometries preserve area

let $\underset{\sim}{f}$ be an isometry between surfaces X and Y

- Y can be parameterised:

$$\underset{\sim}{s} = \underset{\sim}{f} \circ \underset{\sim}{r} \; : \; U \to Y$$

Isometry $\Rightarrow$ $|\Gamma'(t)|^2 = |\hat{\Gamma}'(t)|^2$ ★

$$|\Gamma'(t)|^2 = |\underset{\sim}{r}_u u'(t) + \underset{\sim}{r}_v v'(t)|^2 = E\, u'(t)^2 + 2F\, u'(t)v'(t) + G\, v'(t)^2$$

1st Fundamental Forms

where
$E = \underset{\sim}{r}_u \cdot \underset{\sim}{r}_u$
$F = \underset{\sim}{r}_u \cdot \underset{\sim}{r}_v$
$G = \underset{\sim}{r}_v \cdot \underset{\sim}{r}_v$

Similarly, $|\hat{\Gamma}'(t)|^2 = \hat{E} u'^2 + 2\hat{F} u'v' + \hat{G} v'^2$

$$\hat{E} = \underset{\sim}{s}_u \cdot \underset{\sim}{s}_u, \quad \hat{F} = \underset{\sim}{s}_u \cdot \underset{\sim}{s}_v, \quad \hat{G} = \underset{\sim}{s}_v \cdot \underset{\sim}{s}_v$$

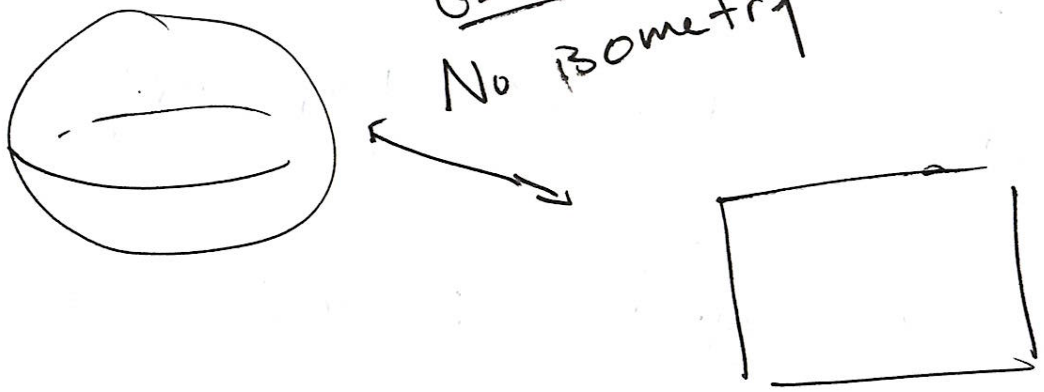
★ holds for all paths : path w/ $u' = 1,\ v' = 0 \Rightarrow E = \hat{E}$
" " " $u' = 0,\ v' = 1 \Rightarrow G = \hat{G}$ $\Rightarrow F = \hat{F}$ also

Surface area: $|\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v| = \sqrt{EG - F^2}$ (PS 7)

$\therefore$ $|\underset{\sim}{s}_u \wedge \underset{\sim}{s}_v| = \sqrt{\hat{E}\hat{G} - \hat{F}^2} = \sqrt{EG - F^2} = |\underset{\sim}{r}_u \wedge \underset{\sim}{r}_v|$ $\therefore$ area preserved.

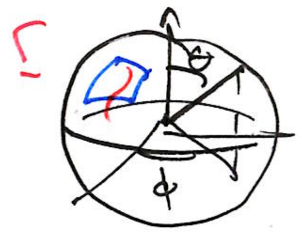
Gauss

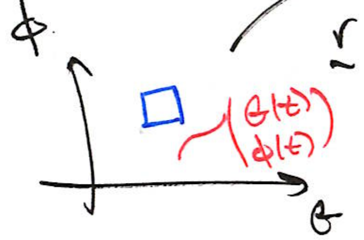
No Isometry



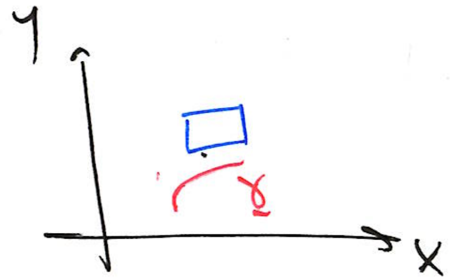
PS 7

$$\begin{cases} x = \frac{1}{n}\sqrt{c - 2n\cos\theta}\,\sin(n\phi) \\ y = \rho_0 - \frac{1}{n}\sqrt{c-2n\cos\theta}\,\cos(n\phi) \end{cases}$$





$$\underset{\sim}{r} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$



$$\underset{\sim}{s} = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ 0 \end{pmatrix}$$

Area: $\quad |\underset{\sim}{r}_\phi \wedge \underset{\sim}{r}_\theta| = \sin\theta$

$$|\underset{\sim}{s}_\theta \wedge \underset{\sim}{s}_\phi| = \left|\frac{\partial(x,y)}{\partial(\theta,\phi)}\right| \overset{?}{=} \sin\theta$$

Lengths: $\quad \underset{\sim}{\gamma} = \begin{pmatrix} x(\theta(t),\phi(t)) \\ y(\theta(t),\phi(t)) \end{pmatrix} \rightarrow |\underset{\sim}{\gamma}'(t)|^2 = \dfrac{(c-2n\cos\theta)^2\phi'(t)^2 + \sin^2\theta\,\theta'(t)^2}{c-2n\cos\theta}$

$$\neq$$

$$\underset{\sim}{\Gamma} = \underset{\sim}{r}(\theta(t),\phi(t)) \rightarrow |\underset{\sim}{\Gamma}'(t)|^2 = \sin^2\theta\,\phi'(t)^2 + \theta'(t)^2$$