C4.3 Functional Analytic Methods for PDEs
Lecture 1

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What is this course about?

- We will be concerned with linear elliptic equations of the form

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

  - ⋆ Ω: a domain in $\mathbb{R}^n$,
  - ⋆ $u : \Omega \to \mathbb{R}$ is the unknown,
  - ⋆ $f : \Omega \to \mathbb{R}$ is a given source,
  - ⋆ $a_{ij} : \Omega \to \mathbb{R}$ are given coefficients with $a_{ij} = a_{ji}$.
  - ⋆ repeated indices are summed from 1 to n, i.e.

$$\partial_i(a_{ij}\partial_j u) = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u).$$

- Linearity: L is linear in the sense that $L(\alpha u + v) = \alpha Lu + Lv$.
- Ellipticity: L is elliptic in the sense that the coefficient matrix $(a_{ij})_{i,j=1}^n$ is positive definite.
- Boundary condition: ignored at the moment.

What is this course about?

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- We will deal with the functional analytic aspects of $(\dagger)$:
  - ⋆ In what functional space should one look for the solutions u?
  - ⋆ In what functional space should one give the sources f?
  - ⋆ In those spaces, is $(\dagger)$ solvable?
  - ⋆ In those spaces, what other properties of solutions does one have?
- We will NOT be concerned with
  - ⋆ Solving for solutions of $(\dagger)$ in closed form.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Classical solutions:
  - ⋆ $u \in C^2(D)$: u has continuous second derivative in D.
  - ⋆ $f \in C(D)$: f is continuous in D.
  - ⋆ $\Delta : C^2(D) \to C(D)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

- Issue 1: Non-existence. The Poisson equation $(\star)$ has no classical solutions for some $f \in C(D)$, e.g.

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \frac{5 - 4\log(x^2 + y^2)}{(1 - \log(x^2 + y^2))^{3/2}}.$$

For this function f, all 'reasonable' solutions are of the form

$$u(x,y) = (x^2 - y^2)(1 - \log(x^2 + y^2))^{1/2} + \text{ an analytic function.}$$

These do not have continuous second derivative at $(0,0)$.

Example 1: The Poisson equation in 2D

$$-\Delta u := -\partial_x^2 u - \partial_y^2 u = f \text{ in the unit disk } D \subset \mathbb{R}^2. \qquad (\star)$$

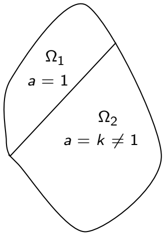
- Issue 2: In some applications, such as heat or electricity conduction on a plate, the source f is not continuous. For example, heat may be supplied only on part of the plate D. In such cases, f is at best piecewise continuous. Naturally the solutions u are no longer in C^2.

Example 2: An equation from material sciences

$$Lu := -\mathrm{div}(a\nabla u) = f \text{ in } \Omega \subset \mathbb{R}^3. \qquad (\star\star)$$

- A composite material occupies a region $\Omega = \Omega_1 \cup \Omega_2$, where each subregion models a different constituent material. The coefficient a thus assumes different values on these subregions, say

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ k \neq 1 & \text{if } x \in \Omega_2. \end{cases}$$



Example 2: An equation from material sciences

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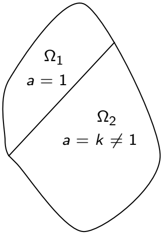

- Issue 1: As a is discontinuous, IF u is smooth, the vector $a\nabla u$ does not have to be continuous and thus the meaning of $\mathrm{div}(a\nabla u)$ is not clear.
- Issue 2: If we instead requires that $a\nabla u$ be continuous, then ∇u may be discontinuous, and so u may not be twice differentiable.

Conclusion

$$Lu := -\partial_i(a_{ij}\partial_j u) + l.o.t. = f \text{ in } \Omega. \qquad (\dagger)$$

- There is a need to consider (generalised/weak) solutions which are not twice differentiable.
- There is a need to consider functions whose (generalised/weak) derivatives are discontinuous.
- GOAL: Treat ($\dagger$) in Sobolev spaces $W^{1,p}$, i.e. space of functions which has first derivatives belonging to L^p.
- Agenda: L^p spaces $\rightsquigarrow$ $W^{1,p}$ spaces $\rightsquigarrow$ Treatment of ($\dagger$).

Tentative plan

- Lebesgue spaces (Chapter 1): Lectures 1-4
- Sobolev spaces (Chapter 2): Lectures 5-7.
- Embedding theorems (Chapter 3): Lecture 8-10.
- Linear elliptic equations in divergence form (Chapter 4): Lecture 11-16.

General expectation

- This course warms up rather casually with L^p theory which many of you are familiar with if you took Part A integration or the equivalence elsewhere, but the pace picks up quickly around end of W3 onwards. I'll try to be as inclusive as possible.
- Do read ahead the lecture notes.
- Though most of the materials in the lecture notes will be discussed in lectures, I may decide occasionally to go over certain topics rather briefly and use the lecture time to discuss something else which is not in the lecture notes. Those either complement what's in the lecture notes, or along the line of exam questions, etc.
- It's highly recommended to consult the various texts given in the lecture notes.

Outline for the rest of the lecture

- Definition of Lebesgue spaces $L^p(E)$.
- Hölder's and Minkowski's inequalities.
- Completeness of Lebesgue spaces – Riesz-Fischer's theorem.
- Converse to Hölder's inequality.
- Duals of Lebesgue spaces.

Lebesgue spaces $L^p(E)$ with $1 \le p < \infty$

- E: a measurable subset of $\mathbb{R}^n$,
- $1 \le p < \infty$, define

$$\mathcal{L}^p(E) = \Big\{ f : E \to \mathbb{R} \Big| \ f \text{ is measurable on } E \text{ and } \int_E |f|^p \, dx < \infty \Big\}.$$

- Define $L^p(E)$ as $\mathcal{L}^p(E)/\sim$ where

$$f \sim g \text{ if } f = g \text{ a.e. in } E.$$

Lebesgue spaces $L^\infty(E)$

- E: a measurable subset of $\mathbb{R}^n$,
- For a measurable $f : E \to \mathbb{R}$, define the essential supremum of f on E by

$$\operatorname*{ess\,sup}_E f = \inf\{c > 0 : f \leq c \text{ a.e. in } E\}.$$

  When $\operatorname{ess\,sup}_E |f| < \infty$, we say f is essentially bounded on E.
- $\mathcal{L}^\infty(E)$ is defined as the set of all essentially bounded measurable functions on E.
- $L^\infty(E)$ is defined as $\mathcal{L}^\infty(E)/\sim$.

Some conventions

- Unless otherwise stated, our functions are real-valued.
- When E is clear, we will simply write L^p in place of $L^p(E)$.
- For simplicity, we will frequently refer to elements of $L^p(E)$ as functions rather than equivalent classes of functions. When there is a need to speak of a representative in an equivalent class of functions, we will make it clear.
- We will use $L^p_{loc}(E)$ to refer to the set of functions f such that, for every compact subset K of E, the restriction of f to K belongs to $L^p(K)$.

$L^p(E)$ is a normed vector space for $1 \leq p \leq \infty$

The following results were proven in Integration:

- The space $L^p(E)$ is a vector space.
- If we define

  $$\|f\|_{L^p(E)} = \left\{ \int_E |f|^p \, dx \right\}^{1/p} \text{ for } 1 \leq p < \infty,$$

  and

  $$\|f\|_{L^\infty(E)} = \operatorname*{ess\,sup}_E |f|,$$

  then $L^p(E)$ is a normed vector space with these norms for $1 \leq p \leq \infty$.

Recap

Recall that $(X, \|\cdot\|)$ is a normed vector space if

- ⋆ X is a vector space
- ⋆ $\|\cdot\|$ maps X into $[0, \infty)$ and satisfies
  - ▷ $\|x\| = 0$ if and only if $x = 0$.
  - ▷ $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}, x \in X$.
  - ▷ $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

$L^p(E)$ is a normed vector space for $1 \leq p \leq \infty$

The following results were proven in Integration:

- In particular, we have

Theorem (Minkowski's inequality)

If $1 \leq p \leq \infty$, *then* $\|f + g\|_{L^p(E)} \leq \|f\|_{L^p(E)} + \|g\|_{L^p(E)}$.

- The proof of the above uses the following important inequality:

Theorem (Hölder's inequality)

If $1 \leq p, p' \leq \infty$ *are such that* $\frac{1}{p} + \frac{1}{p'} = 1$, *then*
$\|fg\|_{L^1(E)} \leq \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}$.

$L^p(E)$ is a Banach space $1 \le p \le \infty$

The following result was touched upon in Integration:

Theorem (Riesz-Fischer's theorem)

If $1 \le p \le \infty$, then $L^p(E)$ is a Banach space with norm $\| \cdot \|_{L^p(E)}$.

Recall that a normed vector space is a Banach space if it is complete with respect to its norm, i.e. all Cauchy sequences converge.

Proof of Riesz-Fischer's theorem

- Suppose that (f_k) is a Cauchy sequence in L^p. We need to show that f_k converges in L^p to some $f \in L^p$.
- Case 1: $p = \infty$.
  - ⋆ For every k, m, there exists a set $Z_{k,m}$ of zero measure such that

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z_{k,m}.$$

  - ⋆ Let $Z = \cup_{k,m} Z_{k,m}$. Then Z has zero measure and

$$|f_k - f_m| \leq \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z \text{ for all } k \text{ and } m.$$

  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.

Proof of Riesz-Fischer's theorem

- Case 1: $p = \infty$...
  - ⋆ So f_k converges uniformly in $E \setminus Z$ to some measurable function $f : E \setminus Z \to \mathbb{R}$. Extend f to all of E by letting $f = 0$ in Z.
  - ⋆ Now, for any k, we have

$$|f_k - f| \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty} \text{ in } E \setminus Z.$$

  - ⋆ As Z has measure zero, this means

$$\|f_k - f\|_{L^\infty} \leq \sup_{m \geq k} \|f_k - f_m\|_{L^\infty}.$$

  - ⋆ Since $f_k \in L^\infty$, it follows from Minkowski's inequality that $f \in L^\infty$. Also, sending $k \to \infty$ in the above inequality also shows that $\|f_k - f\|_{L^\infty} \to 0$, i.e. f_k converges to f in L^∞.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \le p < \infty$.
  - ⋆ We have

$$|\{x \in E : |f_k(x) - f_m(x)| > \varepsilon\}| \le \frac{1}{\varepsilon^p} \int_E |f_k(x) - f_m(x)|^p = \frac{1}{\varepsilon^p} \|f_k(x) - f_m(x)\|_{L^p}^p \stackrel{k,m\to\infty}{\longrightarrow} 0.$$

    This means that the sequence (f_k) is Cauchy in measure.
  - ⋆ A result from Integration then asserts that (f_k) converges in measure, and hence it has a subsequence, say (f_{k_j}), which converges a.e. in E to some function f. To conclude, we show that $f \in L^p$ and $f_k \to f$ in L^p.

Proof of Riesz-Fischer's theorem

- Case 2: $1 \leq p < \infty$...
  - ⋆ Fix some $\delta > 0$, then, for large k and j,

$$\int_E |f_{k_j} - f_k|^p \, dx = \|f_{k_j} - f_k\|_{L^p}^p \leq \delta^p.$$

  - ⋆ Sending $j \to \infty$ and using Fatou's lemma, we get

$$\int_E |f - f_k|^p \, dx \leq \liminf_{j \to \infty} \int_E |f_{k_j} - f_k|^p \, dx \leq \delta^p.$$

  - ⋆ So we have $\|f - f_k\|_{L^p} \leq \delta$ for large k. By Minkowski's inequality, this implies that $f \in L^p$. As δ is arbitrary, this also gives $f_k \to f$ in L^p, as desired.

Dual space of $L^p(E)$

Recall that for a (real) normed vector space X, the dual of X, denoted as X^*, is the Banach space of bounded linear functional $T : X \to \mathbb{R}$, equipped with the dual norm

$$\|T\|_* = \sup \|Tx\|.$$

Theorem (Riesz' representation theorem)

Let E be measurable, $1 \leq p < \infty$ and $p' = \frac{p}{p-1}$. Then there is an isometric isomorphism $\pi : (L^p(E))^* \to L^{p'}(E)$ such that

$$Tg = \int_E \pi(T) g \, dx \text{ for all } g \in L^p(E) \text{ and } T \in (L^p(E))^*.$$

Dual space of $L^p(E)$

Theorem (Riesz' representation theorem)

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- Note the similarity of the above and Riesz' representation theorem for Hilbert spaces. In particular, observe the connection when $p = 2$.
- The theorem is false for $p = \infty$. The dual of $L^\infty(E)$ is strictly bigger than $L^1(E)$. In other words, there exists a linear functional T on $L^\infty(E)$ for which there is no $f \in L^1(E)$ satisfying

$$Tg = \int_E fg \, dx \text{ for all } g \in L^\infty(E).$$