C4.3 Functional Analytic Methods for PDEs
Lecture 2

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In the last lecture

- Definition of Lebesgue spaces.
- Holder's and Minkowski's inequalities
- Completeness of Lebesgue spaces.
- Duals of Lebesgue spaces.

This lecture

- Duals of Lebesgue spaces (cont.).
- L^2 as a Hilbert space.
- Density of simple functions for Lebesgue spaces.
- Separability of Lebesgue spaces.

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- Recall that for a (real) normed vector space X, the dual of X, denoted as X^*, is the Banach space of bounded linear functional $T : X \to \mathbb{R}$, equipped with the dual norm

$$\|T\|_* = \sup \|Tx\|.$$

- $(L^p(E))^* = L^{p'}(E)$ for $1 \leq p < \infty$.
- Consider $p = \infty$. Let $T_k \in (L^\infty(\mathbb{R}))^*$ given by $T_k g = \frac{1}{k}\int_0^k g\,dx$. Then, for every $g \in L^\infty(\mathbb{R})$, $(T_k g) \in \ell^\infty$.
- Let $L \in (\ell^\infty)^*$ be such that

$$L((x_k)) = \lim_{k\to\infty} x_k \text{ provided } (x_k) \text{ is convergent.}$$

  Such L exists by the Hahn-Banach theorem.
- Define $Tg = L((T_k g))$ for all $g \in L^\infty(\mathbb{R})$. It is easy to check that $T \in (L^\infty(\mathbb{R}))^*$.

$(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$

- We claim that there is no $f \in L^1(\mathbb{R})$ such that
$$Tg = \int_{\mathbb{R}} fg\,dx \text{ for all } g \in L^\infty(\mathbb{R}).$$
- Suppose by contradiction that such f exists. Fix some $m > 0$ and let $g_1(x) = sign(f(x))\chi_{(0,m)}(x)$. Then, as $|g_1| \leq \chi_{(0,m)}$, we have for $k > m$ that $|T_k g_1| \leq \frac{m}{k}$. It follows that
$$\int_0^m |f|\,dx = Tg_1 = L((T_k g_1)) = \lim_{k\to\infty} \frac{m}{k} = 0.$$
As m is arbitrary, we thus have $f = 0$ a.e. in $(0,\infty)$.
- On the other hand, with $g_2 = \chi_{(0,\infty)}$, we have $T_k g_2 = 1$ and so
$$0 = \int_0^\infty f\,dx = Tg_2 = L((T_k g_2)) = \lim_{k\to\infty} 1 = 1,$$
which is absurd.

Converse to Hölder's inequality

Proposition (Converse to Hölder's inequality)

Let E be measurable, and f be measurable on E. If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f\|_{L^p(E)} = \sup\Big\{ \int_E fg\,dx : g \in L^{p'}(E), \|g\|_{L^{p'}(E)} \leq 1 \text{ and } fg \text{ is integrable on } E \Big\}.$$

Note: We do not presume that $f \in L^p(E)$.

Proof of Converse to Hölder's inequality

- Will only present the case $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ need some justification; see notes.
- Let

$$\alpha = \sup \Big\{ \int_E fg\, dx : \|g\|_{L^{p'}} \leq 1, fg \in L^1(E) \Big\} \in [0, \infty].$$

  By Hölder's inequality, we have $\alpha \leq \|f\|_{L^p}$. So it suffices to show $\alpha \geq \|f\|_{L^p}$.
- If $\|f\|_{L^p} = 0$, we are done. Assume henceforth that $\|f\|_{L^p} > 0$.

Proof of Converse to Hölder's inequality

- Case 1: $0 < \|f\|_{L^p} < \infty$.
  In this case, we test the definition of α using
  $$g_0(x) = \frac{sign(f(x))|f(x)|^{p-1}}{\|f\|_{L^p}^{p-1}}.$$
  - ⋆ We have, as $p' = \frac{p}{p-1}$,
  $$\int_E |g_0|^{p'}\,dx = \frac{1}{\|f\|_{L^p}^p}\int_E |f|^p\,dx = 1.$$
  - ⋆ Next,
  $$\int_E |f|\,|g_0|\,dx = \frac{1}{\|f\|_{L^p}^{p-1}}\int_E |f|^p\,dx < \infty.$$
  - ⋆ So by the definition of α,
  $$\alpha \geq \int_E f\,g_0\,dx = \frac{1}{\|f\|_{L^p}^{p-1}}\int_E |f|^p\,dx = \|f\|_{L^p}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$.
  In this case, we need to show that $\alpha = \infty$.
  - ⋆ Consider a truncation of $|f|$ given by
  $$f_k(x) = \begin{cases} \min(|f|(x), k) & \text{if } x \in E \text{ and } |x| \leq k, \\ 0 & \text{otherwise.} \end{cases}$$
    Note that we are truncating both in the domain and in the range: $f_k(x) = \min(|f|(x), k)\chi_{E \cap \{|x| \leq k\}}(x)$.
  - ⋆ It is clear that $f_k \in L^p(E)$. Also, by Lebesgue's monotone convergence theorem,
  $$\|f_k\|_{L^p}^p = \int_E |f_k|^p \, dx \to \int_E |f|^p \, dx = \infty.$$
    In addition, by Case 1,
  $$\|f_k\|_{L^p} = \sup\Big\{ \int_E f_k \, g \, dx : \|g\|_{L^{p'}} \leq 1, f_k g \in L^1(E)\Big\}.$$

Proof of Converse to Hölder's inequality

- Case 2: $\|f\|_{L^p} = \infty$...
  - ⋆ In fact, the proof in Case 1 shows that the function $g_k = \frac{|f_k|^{p-1}}{\|f_k\|_{L^p}^{p-1}} \geq 0$ satisfies $\|g_k\|_{L^{p'}} = 1$, $f_k g_k \in L^1(E)$ and
$$\|f_k\|_{L^p} = \int_E f_k\, g_k\, dx.$$
  - ⋆ As $|f| \geq f_k \geq 0$, It follows that, as
$$\int_E |f| g_k\, dx \geq \int_E f_k\, g_k\, dx = \|f_k\|_{L^p} \to \infty.$$
  - ⋆ Letting $\tilde{g}_k(x) = sign(f(x)) g_k(x)$, we then have $\|\tilde{g}_k\|_{L^{p'}} = 1$, $f\tilde{g}_k \in L^1(E)$ and so
$$\alpha \geq \int_E f\tilde{g}_k\, dx = \int_E |f|\, g_k\, dx \to \infty.$$
    So $\alpha = \infty$, as desired.

$L^2(E)$ as a Hilbert space

Theorem

The space $L^2(E)$ is a (real) Hilbert space with inner product

$$\langle f, g\rangle = \int_E fg.$$

This means

- (Banach) $L^2(E)$ is a Banach space.
- (Inner product) The map $(f, g) \mapsto \langle f, g\rangle$ from $L^2(E) \times L^2(E)$ into $\mathbb{R}$ satisfies
  - ⋆ (Linearity) $\langle \lambda f_1 + f_2, g\rangle = \lambda\langle f_1, g\rangle + \langle f_2, g\rangle$ for all $\lambda \in \mathbb{R}, f_1, f_2, g \in L^2(E)$,
  - ⋆ (Symmetry) $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in L^2(E)$,
  - ⋆ (Positivity) $\langle f, f\rangle = \|f\|^2_{L^2(E)}$. Hence $\langle f, f\rangle \geq 0$ for all $f \in L^2(E)$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.

Density results for L^p via simple functions

We will show that the following sets are dense in L^p:

- Set of simple functions, for $1 \leq p \leq \infty$.
- Set of 'rational and dyadic' simple functions, for $1 \leq p < \infty$.

Density results for L^p via simple functions

Simple function:

$$\sum_{i=1}^{N} \alpha_i \chi_{A_i} \text{ where } \alpha_i \text{ is a constant and } A_i \text{ is measurable.}$$

Theorem

Let $1 \le p \le \infty$. The set of all p-integrable simple functions is dense in $L^p(E)$.

Density results for L^p via simple functions

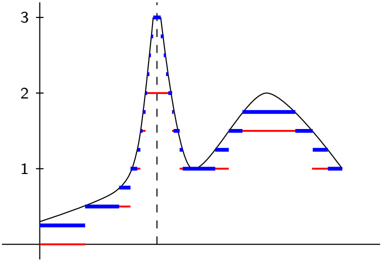
Proof:

- Take $f \in L^p(E)$. We need to construct a sequence (f_k) of p-integrable simple function such that $\|f_k - f\|_{L^p} \to 0$.
- Using the splitting $f = f^+ - f^-$, we may assume without loss of generality that f is non-negative.
- Fact from Integration: If f is a non-negative measurable function, then there exist non-negative simple functions f_k such that $f_k \nearrow f$ a.e.

  Furthermore, if $p < \infty$, then
  - ⋆ $|f_k|^p \leq |f|^p$ and so $f_k \in L^p$;
  - ⋆ As $|f_k - f|^p \leq |f|^p \in L^1$, and so by Lebesgue dominated convergence theorem, $\int_E |f_k - f|^p \, dx \to 0$. So $f_k \to f$ in L^p.

Density results for L^p via simple functions

- When $p = \infty$, the above proof doesn't work as seen. Let us take the proof one step further by recalling how such a sequence f_k can be constructed.
  - ⋆ For each k, one partition the range $[0, \infty]$ into $2^{2k} + 1$ intervals: $J_1^{(k)} = [0, 2^{-k})$, $J_2^{(k)} = [2^{-k}, 2 \times 2^{-k})$, ..., $J_{2^{2k}}^{(k)} = [(2^{2k} - 1) \times 2^{-k}, 2^{2k} \times 2^{-k})$ and $J_{2^{2k}+1}^{(k)} = [2^k, \infty]$.
  - ⋆ f_k is then defined by $f_k(x) = (\ell - 1) \times 2^{-k}$ if $\{f(x) \in J_\ell^{(k)}\}$ for $1 \leq \ell \leq 2^{2k} + 1$.



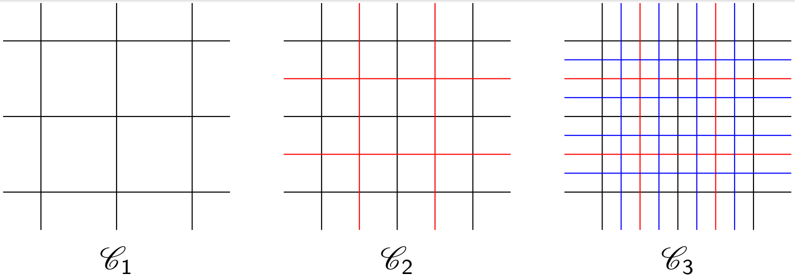
Density results for L^p via simple functions

- When $p = \infty$...
  - ⋆ Aside from the fact that $f_k \nearrow f$, this construction has the property that, in the set $\{f(x) < 2^k\}$, i.e. outside of the set $\{f(x) \in J^{(k)}_{2^{2k}+1}\}$, it holds that
$$|f_k - f| \leq 2^{-k}.$$
  - ⋆ Now as $p = \infty$, f is essentially bounded, i.e. there is an M and a set Z of zero measure such that $f < M$ in $\mathbb{R}^n \setminus Z$. We then redefine f on Z to be zero, i.e. we work with the representative in the 'equivalent class f' which is bounded everywhere by M.
  - ⋆ After this redefinition, we see that $\{f(x) \in J^{(k)}_{2^{2k}+1}\} = \emptyset$ for large k, and so we have $|f_k - f| \leq 2^{-k}$ everywhere for all large k. This means that $f_k \to f$ in L^∞.

Density results for L^p via simple functions

Theorem

Let $1 \le p < \infty$. The set $\mathscr{F}$ of all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes is dense in $L^p(\mathbb{R}^n)$.



$$\mathscr{F} = \Big\{ g = \sum_{i=1}^N r_i \chi_{Q_i} \text{ where } r_i \in \mathbb{Q}, Q_i \in \bigcup_{j=1}^\infty \mathscr{C}_j \Big\}.$$

Density results for L^p via simple functions

Proof:

- We know that the set of p-integrable simple functions is dense in L^p. We also know that $\mathbb{Q}$ is dense in $\mathbb{R}$.
- Thus we only need to show that $\chi_E \in \overline{\mathscr{F}}$.
- By the construction of the Lebesgue measure, every open subset U of $\mathbb{R}^n$ can be written as a countable union of cubes in $\cup \mathscr{C}_i$, say $U = \cup_{i=1}^\infty Q_i$. Then

$$\sum_{i=1}^N \chi_{Q_i} \to \chi_U \text{ in } L^p, \text{ and so } \chi_U \in \overline{\mathscr{F}}.$$

- Now, for every measurable set E of finite measure, the outer regularity of the Lebesgue measure implies that there exist open U_k, $U_k \supset E$ such that $|U_k \setminus E| \to 0$. Then

$$\chi_{U_k} \to \chi_E \text{ in } L^p, \text{ and so } \chi_E \in \overline{\mathscr{F}}.$$

Application: Separability of L^p

Theorem

For $1 \leq p < \infty$, the space $L^p(E)$ is separable, i.e. it has a countable dense subset.

Proof:

- When $E = \mathbb{R}^n$, the result follows from the previous theorem, as $\mathscr{F}$ is countable.
- For general E, let $\tilde{\mathscr{F}}$ be the set of restrictions to E of functions in $\mathscr{F}$. Then $\tilde{\mathscr{F}}$ is countable. We will now show that $\tilde{\mathscr{F}}$ is dense in $L^p(E)$.
  - ⋆ Take $f \in L^p(E)$. Set $f = 0$ in $\mathbb{R}^n \setminus E$. Then $f \in L^p(\mathbb{R}^n)$ and so there exist $f_k \in \mathscr{F}$ such that $f_k \to f$ in $L^p(\mathbb{R}^n)$.
  - ⋆ Let $\tilde{f}_k = f_k|_E \in \tilde{\mathscr{F}}$. Then $\|\tilde{f}_k - f\|_{L^p(E)} \leq \|f_k - f\|_{L^p(E)} \to 0$, so we are done.