# C5.1 Solid Mechanics

## Sheet 0 — MT21

## Background Material

This problem sheet is aimed to be a refresher for some of the key tools that will be used throughout the course. Outline solutions will be published at the end of week 1.

## Einstein's Summation Convention

These exercises are designed to remind you of the Einstein summation convention and to help you to become more fluent with it.

1. Verify the identity

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1}$$

(Convince yourself it is true, e.g. by direct calculation.)

2. Using (1) and the summation convention, prove that the vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}) \tag{2}$$

and that the vector quadruple product

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

**Solution:**

Using the summation convention, we may write:

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \epsilon_{ijk} a_j (\epsilon_{klm} b_l c_m) = \epsilon_{ijk}\epsilon_{lmk} a_j b_l c_m$$

which can be simplified using (1) to

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m = (a_j c_j) b_i - (a_j b_j) c_i = [\mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})]_i$$

as desired.

In a similar way we can write:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\epsilon_{ijk} a_j b_k)(\epsilon_{ilm} c_l d_m) = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k c_l d_m,$$

which immediately gives the desired result.


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3. Using the summation convention, prove the vector calculus identity

$$\text{curl curl } \mathbf{v} = \text{grad div } \mathbf{v} - \text{div grad } \mathbf{v}.$$

Justifying your answer, comment on whether this is the same as would be obtained by substituting $\mathbf{a} = \mathbf{b} = \nabla$ and $\mathbf{c} = \mathbf{v}$ in (2).

**Solution:**

In the now familiar way, we have

$$[\text{curl curl } \mathbf{v}]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l v_m)$$

where $\partial_j(\cdot) = \partial(\cdot)/\partial x_j$. Using (1) we have:

$$[\text{curl curl } \mathbf{v}]_i = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_j(\partial_l v_m) = \partial_j(\partial_i v_j) - \partial_j(\partial_j v_i) = \partial_i(\partial_j v_j) - \partial_j(\partial_j v_i),$$

which is the $i$-th component of the required result.

(Note that here we have treated $\partial_j$ as an operator; in deriving (2) the elements are scalars and so may be commuted as desired. However, care is needed before trying to commute $\partial_j$. Being careless on this point and simply substituting as suggested would lead to a nonsensical identity.)

4. Calculate $\text{curl}(\mathbf{a} \times \mathbf{b})$ using the summation convention and (1).

**Solution:**

We have that

$$[\text{curl}(\mathbf{a} \times \mathbf{b})]_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}a_l b_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(a_l b_m).$$

Using the product rule we therefore have:

$$[\text{curl}(\mathbf{a} \times \mathbf{b})]_i = \partial_j(a_i b_j) - \partial_j(a_j b_i) = a_i(\partial_j b_j) + b_j\partial_j a_i - b_i(\partial_j a_j) - a_j\partial_j b_i,$$

which is the $i$-th component of

$$\mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}.$$


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## Linear Algebra

5. Use the **Cayley–Hamilton Theorem** to find functions $a(n)$ and $b(n)$ such that

$$A^n = a(n)A + b(n)I$$

when

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

and $n$ is an integer. Hence show that

$$\exp A = \begin{pmatrix} e & e^3 - e \\ 0 & e^3 \end{pmatrix}.$$

**Solution:**

The characteristic polynomial is

$$(1-\lambda)(3-\lambda) = 0 = \lambda^2 - 4\lambda + 3.$$

so from the Cayley–Hamilton Theorem we have

$$A^2 = 4A - 3I.$$

A simple induction then gives

$$a(n) = \tfrac{1}{2}(3^n - 1), \quad b(n) = \tfrac{3}{2}(1 - 3^{n-1}).$$

Upon substituting these expressions into the definition of the matrix exponential we find

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}\left[3^n(A-I) + 3I - A\right] = \frac{1}{2}\left[e^3(A-I) + e(3I - A)\right],$$

which immediately gives the desired result.


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6. The **Polar Decomposition Theorem** states that an arbitrary invertible tensor $A$ can be expressed (uniquely) as:

$$A = QU = VQ$$

where $Q$ is an orthogonal tensor and $U$ and $V$ are positive definite symmetric tensors. Here you will give an informal proof of this Theorem, as follows:

(i) Show that the matrices $A^T A$ and $A A^T$ are symmetric and positive definite. [*A matrix B is positive definite if* $\mathbf{x} \cdot (B\mathbf{x}) > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$.]

(ii) Denote the (unique) square root of $A^T A$ and $A A^T$ by $U$ and $V$, respectively — these exist by (i). Show that the matrices $Q = A U^{-1}$ and $R = V^{-1} A$ are orthogonal. [*This proves existence.*]

(iii) Show that the uniqueness of the Polar Decomposition is inherited from the uniqueness of the square roots $U$ and $V$.

(iv) Show that $Q = R$.

**Solution:**

(i) Symmetry is clear by direct calculation. For positive-definiteness, note that: $\mathbf{x} \cdot (A^T A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T A\mathbf{x} = (A\mathbf{x}) \cdot (A\mathbf{x}) = |A\mathbf{x}|^2 > 0$.

(ii) With these definitions, we have $Q^T Q = (U^{-1})^T A^T A U^{-1} = (U^{-1})^T U = I$ (by symmetry of $U$ and $U^{-1}$) so that $Q$ is orthogonal. Similarly for $R$.

(iii) If there were two different polar decompositions, $A = Q_1 U_1 = Q_2 U_2$, say, then we would have $A^T A = U_1 Q^T Q U_1 = U_1^2 = U_2^2$. This is contrary to the assumption of the uniqueness of the square root of $A^T A$, and so uniqueness of the polar decomposition is inherited from that of the square root.

(iv) Now, $A = VR$, and hence we have that

$$A = I(VR) = (RR^T)\,(VR) = R\,(R^T VR) = R\tilde{U}$$

where $\tilde{U} = R^T VR$ is clearly symmetric. However, $U^2 = A^T A = \tilde{U}^T R^T R\tilde{U} = \tilde{U}^2$. Then, by the uniqueness of the square root, we have that $U = \tilde{U}$, and we have shown that $A = RU$. We then immediately have that $R = Q$.


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## Orthogonal Curvilinear Coordinates

7. Consider an orthogonal curvilinear coordinate system $\mathbf{u}(\mathbf{x}) = (u, v, w)$, with $(x, y, z)$ the usual Cartesian coordinate system and unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_w$. Scale factors $h_u$, $h_v$, $h_w$ are defined by

$$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2} \qquad u_i = u, v, w \qquad \text{(no s.c.)}.$$

The gradient of a scalar field is then defined to be

$$\mathrm{grad} f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial u}\mathbf{e}_w$$

while the curl of a vector field is defined via the determinant

$$\mathrm{curl}\,\mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}.$$

(i) Derive the expressions for $\mathrm{curl}\,\mathbf{F}$ and $\mathrm{grad} f$ in spherical polar coordinates $(x, y, z) = (r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, r\cos\theta)$.

(ii) In this coordinate system, find $\mathrm{curl}\,\mathbf{F}$ when

$$\mathbf{F} = (\sinh r)\theta\sin\phi\,\mathbf{e}_r + \frac{\cosh r}{r}\sin\phi\,\mathbf{e}_r + \frac{\cosh r}{r}\frac{\theta}{\sin\theta}\cos\phi\,\mathbf{e}_\phi.$$

(iii) Is there a potential $\varphi$ such that $\mathbf{F} = \mathrm{grad}\,\varphi$? If so find it.

**Solution:**

(i) In spherical polars, $(r, \theta, \phi)$, the scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. We therefore have:

$$\mathrm{grad} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

and

$$\mathrm{curl}\,\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}.$$

(ii) $\mathrm{curl}\,\mathbf{F} = 0$.

(iii) Since $\mathrm{curl}\,\mathbf{F} = 0$ we have that $\mathbf{F} = \nabla\varphi$ for some potential $\varphi$. By inspection of the earlier expression for $\mathrm{grad}\, f$, we see that $\varphi = (\cosh r)\theta(\sin\phi)$.


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