B01 History of Mathematics
Lecture III
The beginnings of calculus, continued
Part 3: Newton and Leibniz

MT 2021 Week 2

Simple 'integrals'

Using the summation rule we can find the quadrature for

$$x^2, \quad x^3, \quad \dots, \quad x^{1/3}, \quad \dots, \quad x^{-4}, \quad \dots$$

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$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \dots$$

but what about

$$(1-x^2)^{1/2} \quad \text{[for a circle]}$$

or

$$(1+x)^{-1} \quad \text{[for a hyperbola]\,?}$$

Enter Newton...

In his own words:

> *In the winter between the years 1664 and 1665 upon reading Dr Wallis's Arithmetica infinitorum and trying to interpole his progressions for squaring the circle, I found out first an infinite series for squaring the circle and then another infinite series for squaring the Hyperbola ...*

Newton extended Wallis' method of interpolation...

Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	?	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	?	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	?	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	?	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	?	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int(1+x)^n dx$. (NB. Newton did not use the notation $\int(1+x)^n dx$.)

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	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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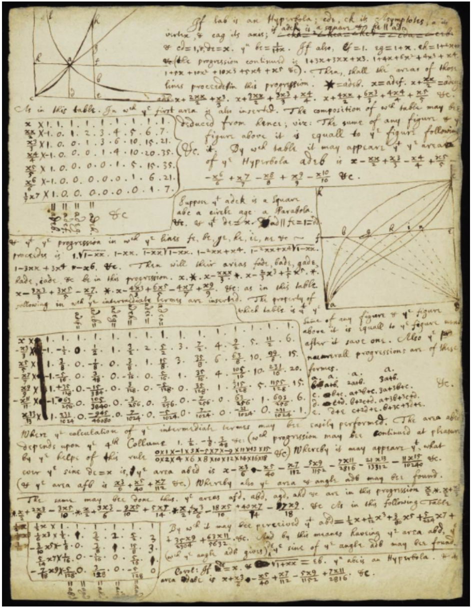
The general binomial theorem

CUL Add. MS 3958.3, f. 72

See *Mathematics emerging*, §8.1.1



Newton's calculus: 1664–5

- rules for quadrature (influenced by Wallis's ideas of interpolation)

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- fluents $x, \; y, \ldots$ (quantities that vary with time t)
- fluxions $\dot{x}, \; \dot{y}, \ldots$ (rate of change of those quantities)

Newton's vocabulary and notation

Newton's calculus 1664–5:

- fluents x, y, ... (quantities that vary with time t)
- fluxions $\dot{x}$, $\dot{y}$, ... (rate of change of those quantities)
- moments o (infinitesimal time in which x increases by $\dot{x}o$)

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propoſ'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2\dot{x}z + a\dot{x}x\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

See the Analise des infiniment petites of the M. de L'Hospital.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ - ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ - y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

E

or

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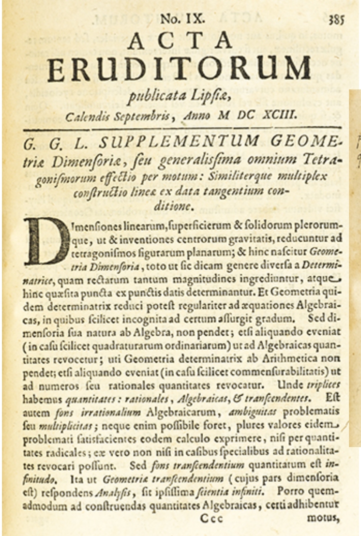
Differentials: du, dv;
integrals: omn. l, later between $S\,\mathrm{l}$ and $\int \mathrm{l}$

Supplementum Geometriae Dimensoriae . . . (1693)



No. IX. 385

ACTA ERUDITORUM

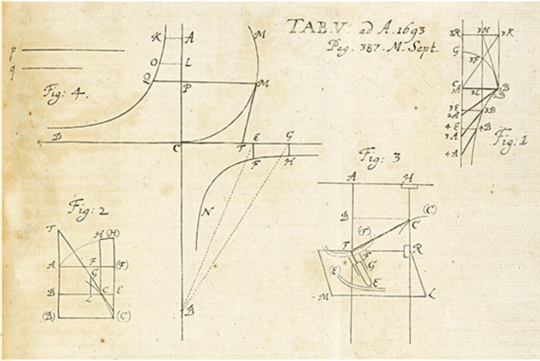
publicata Lipsiæ,

Calendis Septembris, Anno M DC XCIII.

G. G. L. SUPPLEMENTUM GEOMETRIÆ Dimensoriæ, seu generalissima omnium Tetragonismorum effectio per motum: Similiterque multiplex constructio lineæ ex data tangentium conditione.

Dimensiones linearum, superficierum & solidorum plerumque, ut & inventiones centrorum gravitatis, reducuntur ad tetragonismos figurarum planarum; & hinc nascitur *Geometria Dimensoria*, toto ut sic dicam genere diversa a *Determinatrice*, quam rectarum tantum magnitudines ingrediuntur, atque hinc quæsita puncta ex punctis datis determinantur. Et Geometria quidem determinatrix reduci potest regulariter ad æquationes Algebraicas, in quibus scilicet incognita ad certum assurgit gradum. Sed dimensoria sua natura ab Algebra, non pendet; etsi aliquando eveniat (in casu scilicet quadraturarum ordinariarum) ut ad Algebraicas quantitates revocetur; uti Geometria determinatrix ab Arithmetica non pendet; etsi aliquando eveniat (in casu scilicet commensurabilitatis) ut ad numeros seu rationales quantitates revocatur. Unde *triplices* habemus *quantitates: rationales, Algebraicas, & transcendentes.* Est autem *fons irrationalium* Algebraicarum, *ambiguitas* problematis seu *multiplicitas*; neque enim possibile foret, plures valores eidem problemati satisfacientes eodem calculo exprimere, nisi per quantitates radicales; eæ vero non nisi in casibus specialibus ad rationalitates revocari possunt. Sed *fons transcendentium* quantitatum est *infinitudo.* Ita ut *Geometriæ transcendentium* (cujus pars dimensoria est) respondens *Analysis*, sit ipsissima *scientia infiniti.* Porro quemadmodum ad construendas quantitates Algebraicas, certi adhibentur

Ccc motus,



A Supplement to the Geometry of Measurements, or the Most General of all Quadratures to be Effected by a Motion: and likewise the various constructions of a curve from a given condition of the tangent (*Acta eruditorum*, 1693)

Supplementum Geometriae Dimensoriae . . . (1693)

occaſione defungi tandem præſtet, ne intercidant, & ſatis diu iſta, ultra, Horatiani limitis duplum preſſa, Lucinam expectarunt.

Oſtendam autem *problema generale Quadraturarum reduci ad inventionem lineæ datam habentis legem declivitatum*, ſive in qua latera Trianguli characteriſtici aſſignabilis datam inter ſe habeant relationem, deinde oſtendam hanc lineam per motum a nobis excogitatum deſcribi poſſe. Nimirum in omni curva C (C) (*figur. 2*) intelligo *triangulum characteriſticum* duplex: aſſignabile TBC, & inaſſignabile GLC, ſimilia inter ſe. Et quidem *inaſſignabile* comprehenditur ipſis GL LC, elementis coordinatarum CB, CF, tanquam cruribus, & GC, elemento arcûs, tanquam baſi ſeu hypotenuſa. Sed *Aſſignabile* TBC comprehenditur inter axem, ordinatam, & tangentem, exprimitque adeo angulum, quem directio curvæ (ſeu ejus tangens) ad axem vel baſin facit, hoc eſt curvæ declivitatem in propoſito puncto C. Sit jam zona quadranda F(H) comprehenſa inter curvam H(H), duas rectas parallelas FH & (F)(H) & axem F (F) in hoc Axe ſumto puncto fixo A, per A ducatur ad AF normalis AB tanquam axis conjugatus, & in quavis HF (producta prout opus) ſumatur punctum C: ſeu fiat linea nova C(C) cujus hæc ſit natura, ut ex puncto C ducta ad axem conjugatum AB (ſi opus productum) tam ordinata conjugata CB, (æquali AF) quam tangente CT, ſit portio hujus axis inter eas comprehenſa TB, ad BC, ut HF ad conſtantem *a*, ſeu *a* in BT æquetur rectangulo AFH (circumſcripto circa trilineum AFHA). His poſitis ajo rectangulum ſub *a* & ſub E (C) (diſcrimine inter FC & (F)(C) ordinatas curvæ) æquari zonæ F(H); adeoque ſi linea H(H) producta incidat in A, trilineum AFHA figuræ quadrandæ, æquari rectangulo ſub *a* conſtante, & FC ordinata figuræ quadratricis. Rem noſter calculus ſtatim oſtendit, ſit enim AF y; & FH, z; & BT, t; & FC, x; erit $t \sqcap zy:a$, ex hypotheſi: rurſus $t \sqcap y\,dx : dy$ ex natura tangentium noſtro calculo expreſſa. Ergo $a\,dx \sqcap z\,dy$, adeoque $ax \sqcap \int z\,dy \sqcap$ AFHA. Linea igitur C (C) eſt *quadratrix* reſpectu lineæ H(H), cum ipſius C(C) ordinata FC, ducta in *a* conſtantem, faciat rectangulum æquale areæ ſeu ſummæ ordinatarum ipſius H(H) ad abſciſſas debitas AF applicatarum. Hinc cum BT ſit ad AF ut FH ad *a* (ex hypotheſi) deturque relatio ipſius FH ad AF (naturam exhibens figuræ quadrandæ) dabitur ergo & relatio BT

ad

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i.e., integration is reduced to the finding of a curve with a particular tangent — in modern terms, the antiderivative

For Latin-readers: full paper available online

Newton's calculus and Leibniz's calculus compared

Newton (1664–65):	Leibniz (1673–76):
rules for quadrature rules for tangents 'fundamental theorem'	rules for quadrature rules for tangents 'fundamental theorem'
dot notation	differential notation
physical intuition: rates of change	algebraic intuition rules and procedures
PROBLEM: vanishing quantities o	PROBLEM: vanishing quantities du, dv, ...

An elementary introduction to the development of calculus