B01 History of Mathematics
Lecture III
The beginnings of calculus, continued
Part 2: Indivisibles and infinitesimals

MT 2021 Week 2

New methods: indivisibles and infinitesimals

Indivisibles:	geometric objects making up a higher-dimensional object (e.g., points $\rightarrow$ line, lines $\rightarrow$ plane)
Infinitesimal:	arbitrarily small but nonzero quantity

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During the 17th century, both concepts saw much use — despite the fact that they appeared to contradict Euclidean principles

Indivisibles

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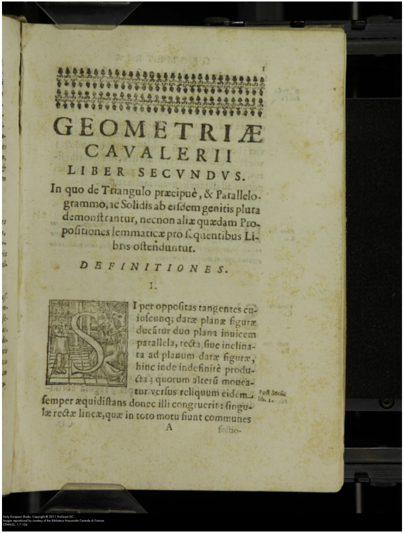
Developed by John Wallis (1616–1703) and others.

Cavalieri's *Geometria*



GEOMETRIÆ
CAVALERII
LIBER SECVNDVS.

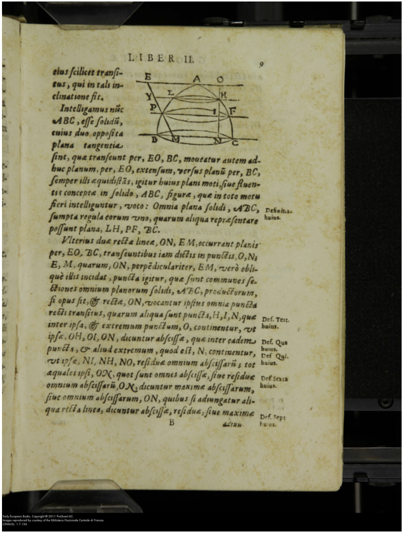
In quo de Triangulo præcipuè, & Parallelogrammo, ac Solidis ab eisdem genitis plura demonstrantur, necnon aliæ quædam Propositiones lemmaticæ pro sequentibus Libris ostenduntur.

DEFINITIONES.

I.

SI per oppositas tangentes cuiuscunq; datæ planæ figuræ ducātur duo plana inuicem parallela, recta siue inclinata ad planum datæ figuræ, hinc inde indefinitè producta; quorum alterū moueatur versus reliquum eidem semper æquidistans donec illi congruerit: singulæ rectæ lineæ, quæ in toto motu fiunt communes

Post Secūd. lib. 1.



eius scilicet transitus, qui in tali inclinatione fit.

Intelligamus nūc, ABC, esse solidū, cuius duo opposita plana tangentia sint, quæ transeunt per, EO, BC, moueatur autem adhuc planum, per, EO, extensum, versus planū per, BC, semper illi æquidistās, igitur huius plani moti, siue fluentis conceptæ in solido, ABC, figuræ, quæ in toto motu fieri intelliguntur, voco: Omnia plana solidi, ABC, sumpta regula eorum vno, quarum aliqua repræsentare possunt plana, LH, PF, BC.

Definit. 2. huius.

Vlterius duæ rectæ lineæ, ON, EM, occurrant planis per, EO, BC, transeuntibus iam dictis in punctis, O, N; E, M, quarum, ON, perpēdiculariter, EM, verò obliquè illis incidat, puncta igitur, quæ sunt communes sectiones omnium planorum solidi, ABC, productorum, si opus sit, & rectæ, ON, vocantur ipsius omnia puncta recti transitus, quarum aliqua sunt puncta, H, I, N, quæ inter ipsa, & extremum punctum, O, continentur, vt ipsæ, OH, OI, ON, dicuntur abscissæ, quæ inter eadem puncta, & aliud extremum, quod est, N, continentur, vt ipsæ, NI, NH, NO, residuæ omnium abscissarū; tot æquales ipsi, ON, quot sunt omnes abscissæ, siue residuæ omnium abscissarū, ON, dicuntur maximæ abscissarum, siue omnium abscissarum, ON, quibus si adiungatur aliqua recta linea, dicuntur abscissæ, residuæ, siue maximæ

Def. Tert. huius.

Def. Qua huius.

Def Qui. huius.

Def. Sexta huius.

Def. Sept huius.

Torricelli's hyperbolic solid (*Opera geometrica*, 1644)

Problema Secundum. 115

Lemma V.

Cuiuſcunque cylindri ghil *intra ſolidum acutum deſcripti (vt in præcedenti figura) ſuperficies ſine baſibus æqualis eſt circulo cuius ſemidiameter ſit linea* df. *nempe ſemiaxis, ſiue ſemilatus verſum ipſius hyperbolæ. Hoc enim in ipſo progreſſu præcedentis lemmatis demonſtratum eſt.*

Theorema.

Solidum acutum hyperbolicum infinitè longum, ſectum plano ad axem erecto, vnà cum cylindro ſuæ baſis, æquale eſt cylindro cuidam recto, cuius baſis diameter ſit latus verſum, ſiue axis hyperbolæ, altitudo verò ſit æqualis ſemidiametro baſis ipſius acuti ſolidi.

Eſto hyperbola cuius aſymptoti *ab*, *ac* angulum rectum contineant; ſumptoq; in hyperbola quolibet puncto *d*, ducatur *dc* æquidiſtans ipſi *ab*, & *dp* æquidiſtans *ac*. Tũ conuertatur vniuerſa figura circa axẽ *ab*. ità vt fiat ſolidum acutum hyperbolicum *ebd*, vna cum cylindro ſuæ baſis *fedc*. Poducatur *ba* in *h*. ita vt *ah* æqualis ſit integro axi, ſiue lateri verſo hyperbolæ. Et circa diametrum *ah* intelligatur circulus erectus ad aſymptoton *ac*: & ſuper baſi *ah* concipiatur cylindrus rectus *acgh*, cuius altitudo ſit *ac*, nempe ſemidiameter baſis acuti ſolidi. Dico ſolidum vniuerſum *febdc*, quanquam ſine fine longum, æquale tamen eſſe cylindro *acgh*.

Accipiatur in recta *ac* quodlibet punctum *i*, & per *i* intelligatur ducta ſuperficies cylindrica *onli* in ſolido acuto

P 2 com-

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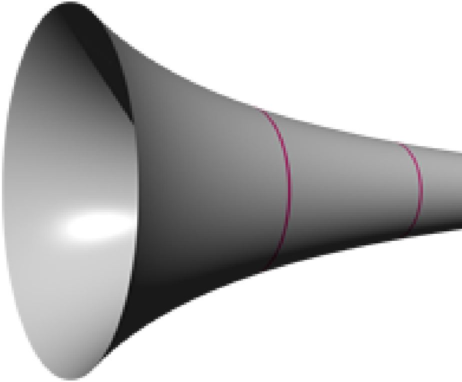


(See *Mathematics emerging*, §3.3.1.)

John Wallis (1616–1703)

Studied at Emmanuel College, Cambridge (BA 1637, MA 1640)

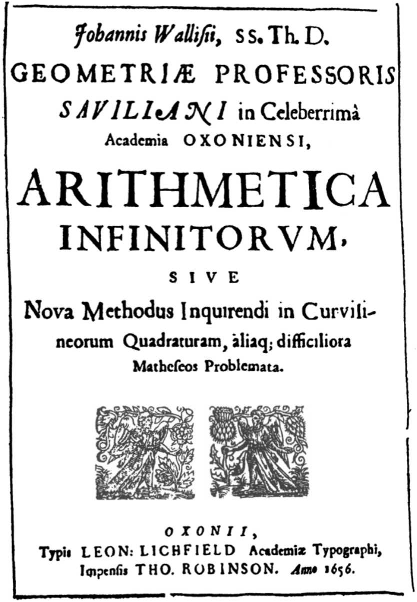
1643–1649: scribe for Westminster Assembly

1644–1645: Fellow of Queens' College, Cambridge

1643–1689: cryptographer to Parliament, then to the Crown

1649–1703: Savilian Professor of Geometry in Oxford

Arithmetica infinitorum



Johannis Wallisii, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academia OXONIENSI,
ARITHMETICA
INFINITORVM,
SIVE
Nova Methodus Inquirendi in Curvilineorum Quadraturam, àliaq; difficiliora Matheseos Problemata.

OXONII,
Typis LEON: LICHFIELD Academiæ Typographi,
Impensis THO. ROBINSON. *Anno* 1656.

John Wallis,
Arithmetica infinitorum
(*The arithmetic of infinitesimals*)
Oxford, 1656

Translation by
Jacqueline A. Stedall
Springer, 2004

Arithmetica infinitorum

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- Investigation of sums of sequences of powers (or ratios of these to a known fixed quantity) — usually decreasing
- Fixed an endpoint, dividing interval into infinite number of arbitrarily small subintervals — these are the 'infinitesimals' of Wallis' title

Wallis and indivisibles

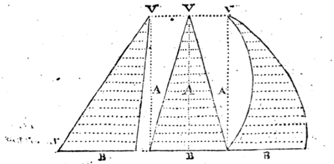
PROP. II. *Theorema.*

SI ſumatur ſeries quantitatum Arithmeticè proportionalium (ſive juxta naturalem numerorum conſecutionem) continuè creſcentium, a puncto vel 0 inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim diſcriminis cauſa erit,) erit illa ad ſeriem totidem maximæ æqualium, ut 1 ad 2.

Nempe, ſi primus terminus ſit 0, ſecundus 1, (nam ſi ſecus, moderatio adhibenda erit,) & ultimus l erit ſumma $\frac{l+1}{2}l$. (erit enim, eo caſu, numerus terminorum $l+1$.) vel, (poſito numero terminorum a, quantuſcunq; ſit terminus ſecundus) $\frac{1}{2}al$.

PROP. III. *Corollarium.*

ERgo, *Triangulum ad Parallelogrammum (ſuper æquali baſe, æquè altum,) eſt ut* 1 *ad* 2.



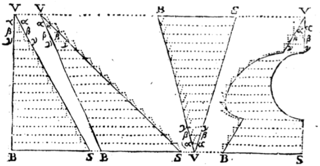
Triangulum

For the triangle … consists of an infinite number of parallel lines in arithmetic proportion …

(See *Mathematics emerging*, §2.4.2.)

Wallis and indivisibles?

ralis initium. Quamvis enim Sectorum illorum numero infinitorum aggregatum, ipſi figuræ lineis recta & Spirali terminatæ, (juxta methodum Indiviſibilium) æquale ponatur; non tamen illud de omnium Arcubus cum ipſa Spirali (propriè dicta) comparatis obtinebit. Tantundem enim eſſet, ac ſi quis, dum infinita numero parallelogramma triangulo inſcripta (aut etiam circumſcripta) toti triangulo VBS æqualia videat, inde



concluderet eorum omnium latera rectæ VS adjacentia (rectæ VB parallela) ipſi VS ſimul æqualia eſſe, vel quæ rectæ VB adjacent (ipſi VS parallela) æqualia ſimul eſſe toti VB. (Quod ſiquando verum eſſe contingat, puta in triangulo iſoſceli, non tamen id univerſaliter concludendum erit.) Atq; hoc quidem eo potius admonendum duxi, quod viderim etiam viros doctos nonnunquam ſpecioſa ejuſmodi veriſimilitudine in lapſum proclives eſſe. Cur autem omiſſa Spirali genuina, ſpuriam hanc peripheriæ comparaverim; cauſa eſt, quod huic poſſim, non autem illi, æqualem peripheriam aſſignare.

PROP. XIV. *Corollarium.*

ET propterea etiam *ſegmenta ſpiralis, a principio ſpiralis exorſa, ſunt ad rectas conterminas, ſicut Parabolæ Diametri interceptæ, ad ordinatim-applicatas.*

For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed about) a triangle, it seems that they equal to complete triangle . . .

(See *Mathematics emerging*, §2.4.2.)

Sums of powers

Wallis' method depended upon the summation rule

$$\sum_{a=0}^{A} a^n \approx \frac{A^{n+1}}{n+1}$$

This was known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n

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This was known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n, but in the 1650s Wallis extended it to negative and fractional n.

Simple 'integrals'

Using the summation rule we can find the quadrature for

$$x^2, \quad x^3, \quad \ldots, \quad x^{1/3}, \quad \ldots, \quad x^{-4}, \quad \ldots$$

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$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \ldots$$

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and

$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \ldots$$

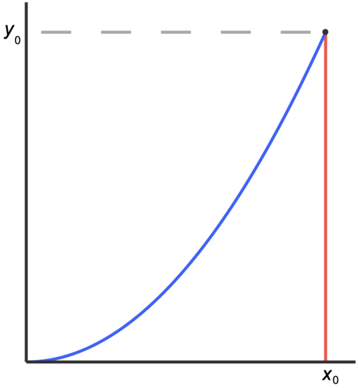
but what about

$$(1-x^2)^{1/2} \quad \text{[for a circle]}$$

or

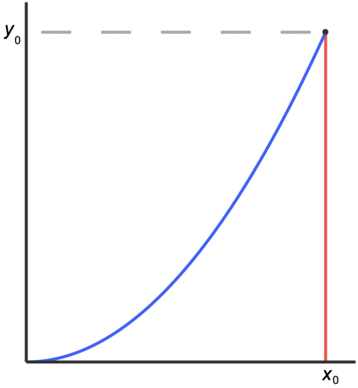
$$(1+x)^{-1} \quad \text{[for a hyperbola]}\,?$$

Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

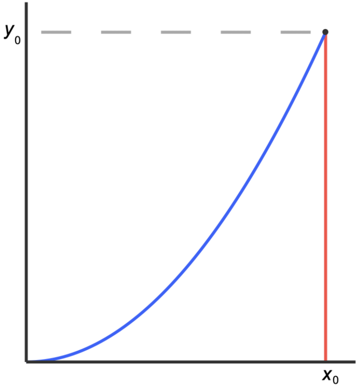
Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

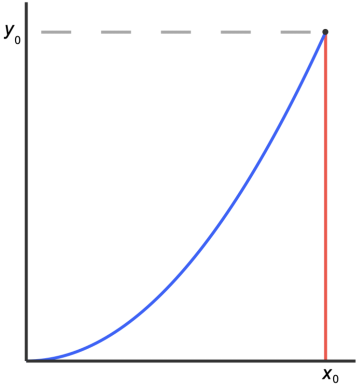
He used the language of ratio, hence sought to calculate the ratio of the area A under the curve to that of the corresponding rectangle $(x_0 y_0)$, which we may think of as the fraction $\frac{A}{x_0 y_0}$

Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?)

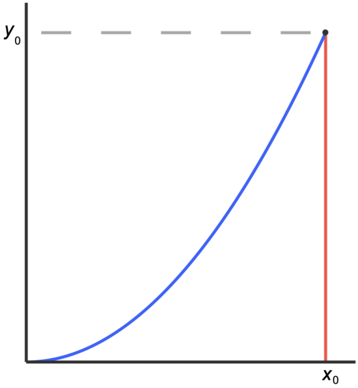
Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?), so

- A is the sum of the values of x^2 as x ranges from 0 to x_0
- $x_0 y_0$ is the sum of as many copies of x_0^2 (?)

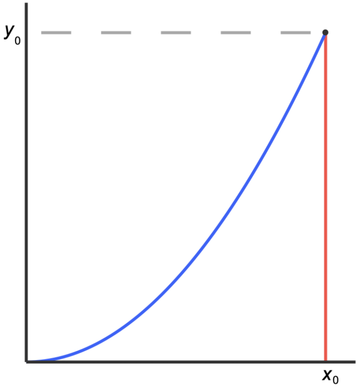
Wallis and the quadrature of a parabola



Break $(0, x_0)$ into n subintervals, suppose that x only takes the values at the endpoints of these, and consider the ratio

$$R = \frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2}$$

Wallis and the quadrature of a parabola



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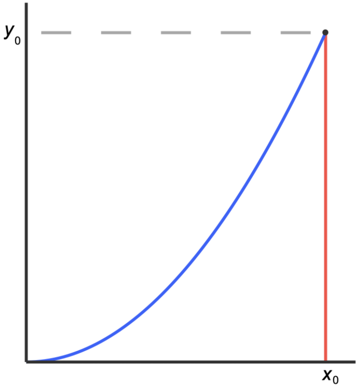
As we make n larger, this ratio will become a closer approximation to $\frac{A}{x_0 y_0}$

Wallis and the quadrature of a parabola



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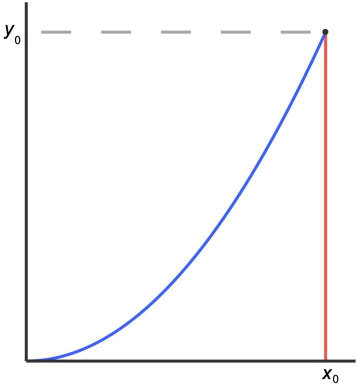
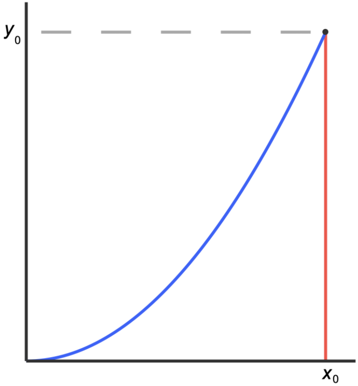
[Note that we are deliberately avoiding the terminology of limits, and that some x_0^2s have been cancelled, thanks to the use of ratios]

Wallis and the quadrature of a parabola



Wallis investigated the cases of small n

Wallis and the quadrature of a parabola



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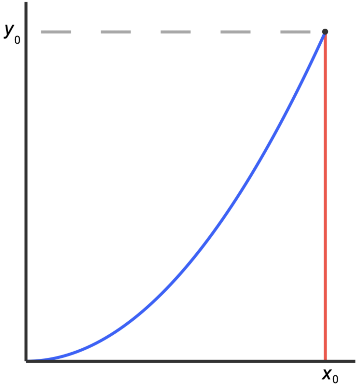
For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2}$$

Wallis and the quadrature of a parabola



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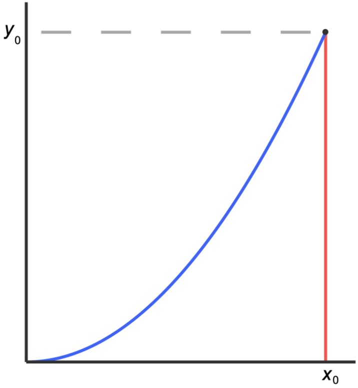
For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2}$$

Wallis and the quadrature of a parabola

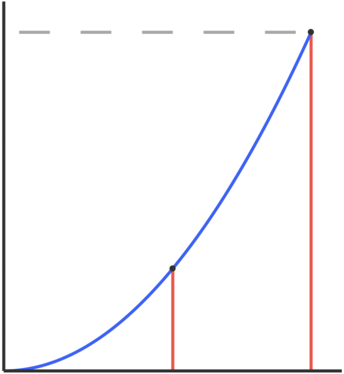


Wallis investigated the cases of small n

For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

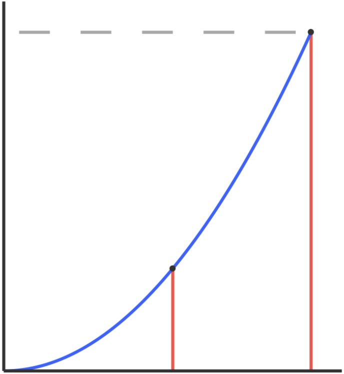
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2}$$

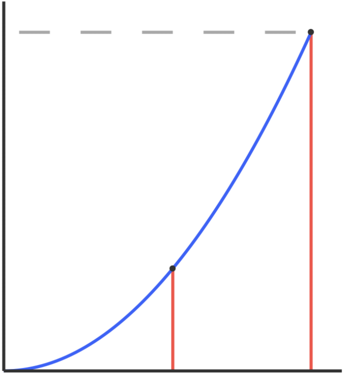
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12}$$

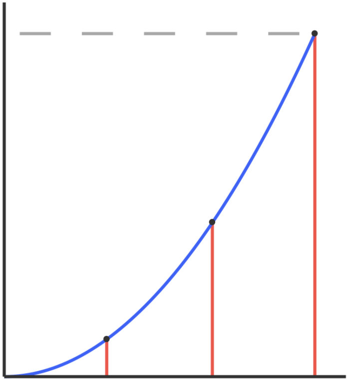
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

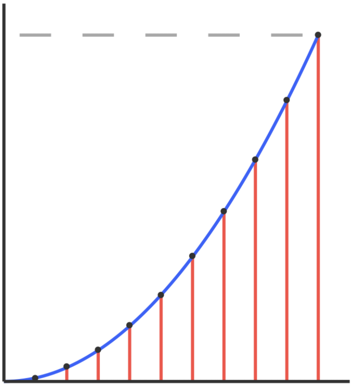
Wallis and the quadrature of a parabola



For $n = 3$ (three red lines),

$$R = \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2}$$
$$= \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$

Wallis and the quadrature of a parabola



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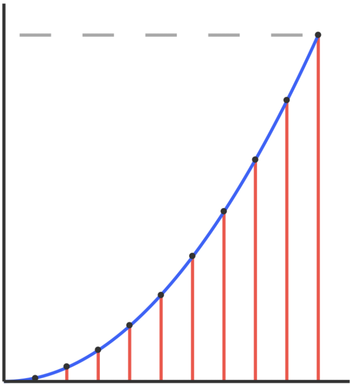

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Wallis called this method of spotting and extending a pattern 'induction' — it was criticised at the time (for example, by Pascal)