BO1 History of Mathematics
Lecture III
The beginnings of calculus, continued
Part 1: Quadrature

MT 2021 Week 2

Summary

Part 1

- *Enri*: a non-Western prelude
- Quadrature (finding areas)

Part 2

- Indivisibles
- Infinitesimals

Part 3

- The contributions of Newton & Leibniz

Seki and *enri*

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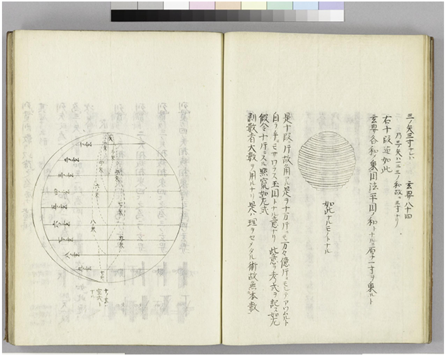

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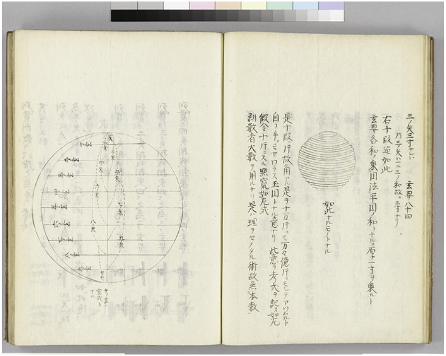

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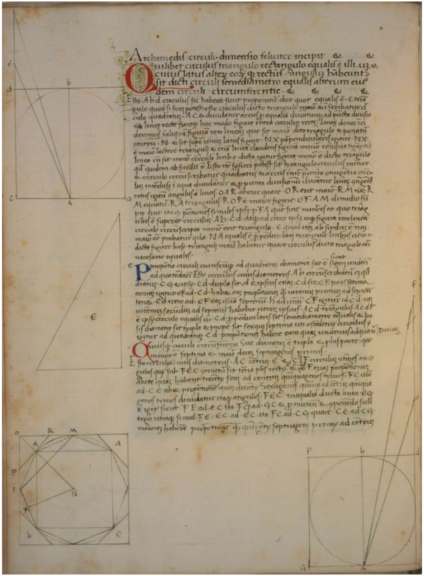
But *enri* was much narrower in scope than calculus

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



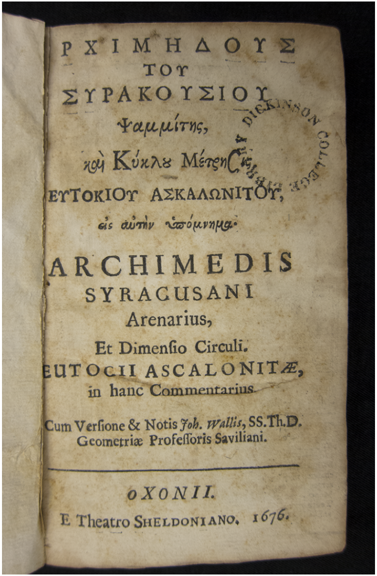
Archimedis circuli dimensio feliciter incipit

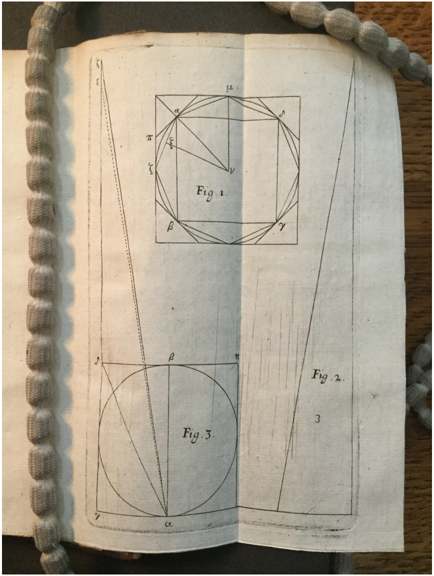
Translated into Latin as *Dimensio circoli* by Jacobus Cremonensis, c. 1450–1460

Illustrated by Piero della Francesca

Available online with other texts by Archimedes

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)





Edition by John Wallis, Oxford, 1676

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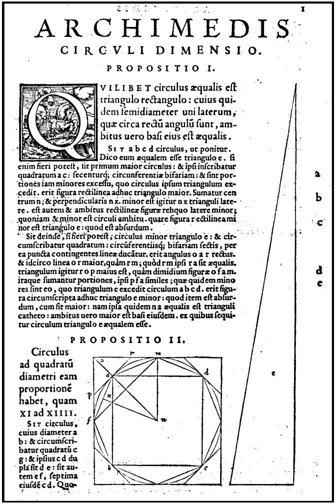


1

ARCHIMEDIS

CIRCVLI DIMENSIO.

PROPOSITIO I.

QVILIBET circulus æqualis eſt triangulo rectangulo: cuius quidem ſemidiameter uni laterum, quæ circa rectũ angulũ ſunt, ambitus uero baſi eius eſt æqualis.

SIT a b c d circulus, ut ponitur. Dico eum æqualem eſſe triangulo e. ſi enim fieri poteſt, ſit primum maior circulus: & ipſi inſcribatur quadratum a c: ſecenturq́; circunferentiæ bifariam: & ſint portiones iam minores exceſſu, quo circulus ipſum triangulum excedit. erit figura rectilinea adhuc triangulo maior. Sumatur centrum n; & perpendicularis n x. minor eſt igitur n x trianguli latere. eſt autem & ambitus rectilineæ figuræ reliquo latere minor; quoniam & minor eſt circuli ambitu. quare figura rectilinea minor eſt triangulo e: quod eſt abſurdum.

Sit deinde, ſi fieri poteſt, circulus minor triangulo e: & circumſcribatur quadratum: circũferentiiſq́; bifariam ſectis, per ea puncta contingentes lineæ ducãtur. erit angulus o a r rectus. & idcirco linea o r maior, quàm r m; quòd r m ipſi r a ſit æqualis. triangulum igitur r o p maius eſt, quàm dimidium figuræ o f a m. itaque ſumantur portiones, ipſi p f a ſimiles; quæ quidem minores ſint eo, quo triangulum e excedit circulum a b c d. erit figura circumſcripta adhuc triangulo e minor: quod item eſt abſurdum, cum ſit maior: nam ipſa quidem n a æqualis eſt trianguli catheto: ambitus uero maior eſt baſi eiuſdem. ex quibus ſequitur circulum triangulo e æqualem eſſe.

PROPOSITIO II.

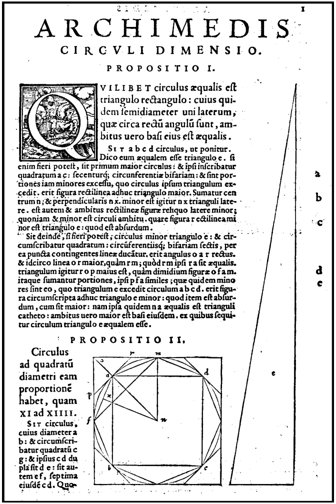
Circulus ad quadratũ diametri eam proportionẽ habet, quam XI ad XIIII.

SIT circulus, cuius diameter a b: & circumſcribatur quadratũ c g: & ipſius c d dupla ſit d e: ſit autem e f, ſeptima eiuſdẽ c d. Quo-

(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



ARCHIMEDIS
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QVILIBET circulus æqualis est triangulo rectangulo: cuius quidem semidiameter uni laterum, quæ circa rectũ angulũ sunt, ambitus uero basi eius est æqualis.

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SIT circulus, cuius diameter a b: & circumscribatur quadratũ c g: & ipsius c d dupla sit d e: sit autem e f, septima eiusdẽ c d. Quo-

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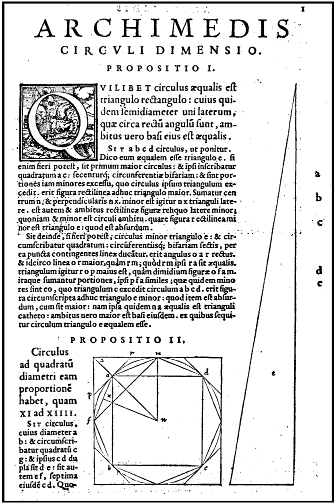
A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

Proof by exhaustion and double contradiction

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



I

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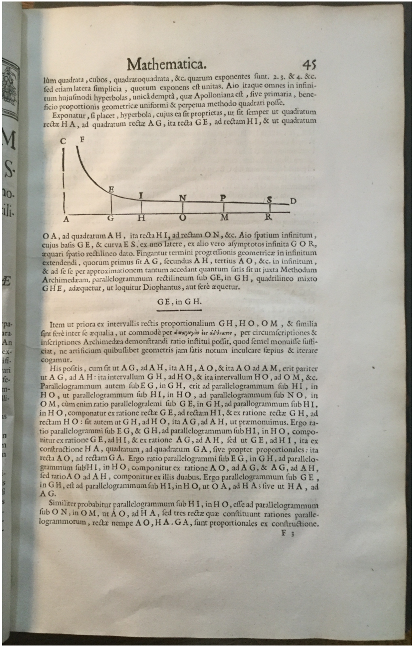
Later: the ratio of the circumference to the diameter is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

Fermat's quadrature of a hyperbola (c. 1636)



Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum

O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

GE, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A. G A, ſunt proportionales ex conſtructione.

F 3

Worked out c. 1636, but only published posthumously in *Varia opera mathematica*, 1679.

In modern terms, this is the curve described by $y = \dfrac{1}{x^2}$.

See *Mathematics emerging*, §3.2.1.

The rectangular (or ‘Apollonian’) hyperbola

In modern notation, $y = \dfrac{1}{x}$

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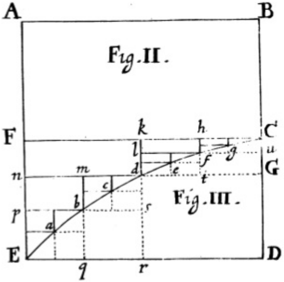
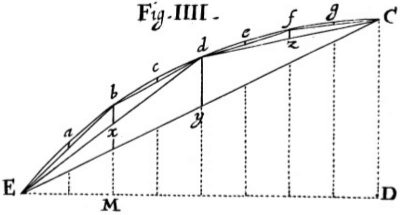
- Problem solved in early 1650s by William Brouncker; published in 1668 in volume 3 of *Philosophical Transactions of the Royal Society*

Brouncker's quadrature of the hyperbola (1668)





To put this into modern terms, take A as the origin, and AB, AE as the x- and y-axes, respectively. Then the diagram represents the area under $\dfrac{1}{1+x}$ from $x=0$ to $x=1$.

(See *Mathematics emerging*, §3.2.2.)

Brouncker's article of 1668

(645) *Numb.34.*

PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola, *by an infinite ſeries of Rational Numbers, together with its Demonſtration, by that Eminent Mathematician, the Right Honourable the Lord Viſcount* Brouncker.

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My Method for Squaring the Hyperbola *is this:*

LEt AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where stands for Multiplication.

Supposing the Reader knows, that EA. $\alpha\zeta$. KH. $\beta\nu$. dθ. $\gamma\kappa$. $\delta\lambda$. $\varepsilon\mu$. CB. &c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmetice proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisij*:)

I say ABCdEA $= \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10}$ &c.

EdCDE $= \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11}$ &c.

EdCyE $= \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10}$ &c.

} *in infinitum.*

For (in Fig. 2, & 3) the Parallelog.

CA $= \frac{1}{1\times2}$

dD $= \frac{1}{2\times3}$	dF $= \frac{1}{3\times4}$
br $= \frac{1}{4\times5}$	bn $= \frac{1}{5\times6}$
fG $= \frac{1}{6\times7}$	fk $= \frac{1}{7\times8}$
aq $= \frac{1}{8\times9}$	ap $= \frac{1}{9\times10}$
cs $= \frac{1}{10\times11}$	cm $= \frac{1}{11\times12}$
et $= \frac{1}{12\times13}$	el $= \frac{1}{13\times14}$
gu $= \frac{1}{14\times15}$	gh $= \frac{1}{15\times16}$
&c.	&c.

And (in Fig. 4.) the Triangl.

EdC $= \frac{1}{2\times3\times4} = \frac{\square dD - \square dF}{2}$

Ebd $= \frac{1}{4\times5\times6} = \frac{\square br - \square bn}{2}$

dfC $= \frac{1}{6\times7\times8} = \frac{\square fG - \square fk}{2}$

Eab $= \frac{1}{8\times9\times10} = \frac{\square aq - \square ap}{2}$

bcd $= \frac{1}{10\times11\times12} = \frac{\square cs - \square cm}{2}$

def $= \frac{1}{12\times13\times14} = \frac{\square et - \square el}{2}$

fgC $= \frac{1}{14\times15\times16} = \frac{\square gu - \square gh}{2}$

&c.

Note.

$\frac{1}{2}$CA = dD + dF

$\frac{1}{2}$dD = br + bn

$\frac{1}{2}$dF = fG + fk

$\frac{1}{2}$br = aq + ap

$\frac{1}{2}$bn = cs + cm

$\frac{1}{2}$fG = et + el

$\frac{1}{2}$fk = gu + gh

&c.

And