MATHEMATICAL INSTITUTE
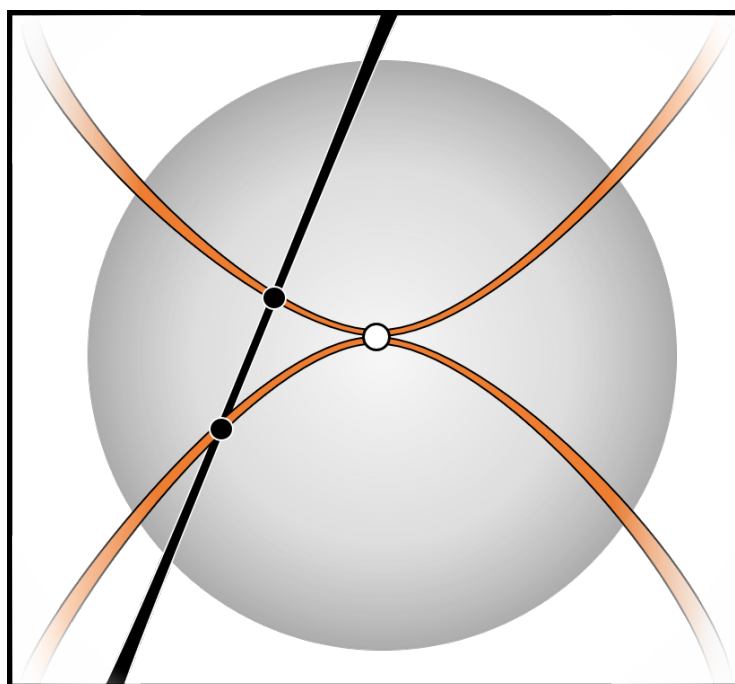
UNIVERSITY OF OXFORD

Computational Mathematics

Student Guide

Michaelmas Term 2021

by

Prof. Nick Trefethen

Notes for the 2021/2022 Course

The course will be in two parts: Part I in Michaelmas Term Weeks 3–8 and Hilary Term Weeks 1–2, and Part II in Hilary Term Weeks 3–8. For Part I, you will attend scheduled practical sessions every fortnight starting in Week 3 Michaelmas Term. The practical sessions are held in the Mathematical Institute, Radcliffe Observatory Quarter.

Each practical session will be run by a demonstrator and may include a short lecture. **You will need to bring your laptop to these sessions.** Please follow the instructions on the Mathematical Institute website

https://www.maths.ox.ac.uk/members/it/software-personal-machines/matlab

to install MATLAB on your machine before the first session. Note that there are around 50 toolboxes available for download, and the only one you will require is the Symbolic Math Toolbox. The default setting is to download all available toolboxes, which may take many hours using a standard broadband connection. To avoid this, you can select just the Symbolic Math Toolbox (i.e., deselect all other toolboxes) in the "Product Selection Window" stage of the installation process.

As an alternative to MATLAB, and in particular if you are a proficient computer user, you could try to use Octave (http://www.octave.org) which uses a very similar command syntax and is often regarded as a MATLAB clone. See Section 1.1.2.

Inevitably Prof. Trefethen will be unable to resist saying a few words now and then about Chebfun, the continuous analogue of MATLAB. If you are curious, see www.chebfun.org.

Time-line   There are four problem sheets, which will be the focus of four corresponding demonstrator sessions, scheduled in weeks 3-4, 5-6, 7-8 of MT and 1-2 of HT, and which correspond to the four chapters of the course.

Weeks	Chapter
3–4 MT	1
5–6 MT	2
7–8 MT	3
1–2 HT	4

Michaelmas term will place many demands on your time, but you need to put in the effort to ensure that you make steady progress and finish Part I of the course by the end of Week 2, Hilary Term.

Assessment   During Part I of the course, you may work collaboratively with others and—as always—you are encouraged to discuss mathematics and your studies with your peers. Your demonstrator will ask you to submit work related to these sessions: chapter 1/2/3 exercises to be turned in at sessions 2/3/4. However, none of the work in Part I will be formally assessed. Instead the material will act as a foundation enabling you to work individually during Part II. This individual work will be assessed and will count towards your Preliminary Examination as described in *Examination Decrees & Regulations*, 2021 and the current *Course Handbook*. (See your college tutor if you have any questions about this aspect of the course.)

Course director   The course director is Prof. Nick Trefethen. The demonstrators will be able to answer most questions; alternatively Prof. Trefethen can be contacted at trefethen@maths.ox.ac.uk.

Course website   This manual and any extra course material (such as lecture notes) can be found on the course website:

https://courses.maths.ox.ac.uk/course/view.php?id=44

Errata list   If errors are found in this manual, they will be corrected and listed on the website above. It would be a good idea to check occasionally, especially if you think you've found an error.

Contents

Chapter 1

Introduction to MATLAB

In this chapter you will learn about:

- Accessing MATLAB,
- using MATLAB as a calculator,
- MATLAB identifiers, constants and functions,
- using the MATLAB help system,
- making `.m` file scripts,
- plotting simple line graphs.

1.1 What is MATLAB?

MATLAB is often described as a problem solving environment. It is a programming language and a set of tools for solving mathematical problems.

The name MATLAB was originally a contraction of "Matrix Laboratory" and indeed MATLAB's core strength is numerical computing involving matrices, vectors and linear algebra. It also has extensive plotting and graphics routines.

1.1.1 What is the Symbolic Math Toolbox?

The Symbolic Math Toolbox is an add-on for MATLAB which adds symbolic as opposed to numerical computing abilities. The strength of such a system is the ability to manipulate *algebraic* expressions. The Symbolic Math Toolbox can be used to solve algebraic equations, factorize polynomials and simplify rational expressions. It can also solve some differential equations. It can perform many of the standard operations of analysis such as evaluating sums, limits, derivatives and integrals. Under the hood, the Symbolic Math Toolbox is powered by a computer algebra system called MuPAD.

The combination of MATLAB and the Symbolic Math Toolbox allows the user to mix symbolic and numerical computing within the MATLAB environment.

1.1.2 What is Octave?

Some people are concerned about issues surrounding computing. Should software be freely available? If these matters are important to you, you might want to consider Octave (http://www.octave.org) as an alternative to MATLAB. It is Free Software (also known as Open Source Software) which means you can use it without restriction, study its source code, improve it, and share it with others as you wish. None of these things are true of MATLAB.

OctSymPy (see http://github.com/cbm755/octsympy) is an add-on package that adds symbolic computing features to Octave (it is analogous to the Symbolic Math Toolbox for MATLAB). OctSymPy is new software developed right here at Oxford!

The syntax of commands in Octave is almost the same as in MATLAB, and the differences are well documented. But having said that, this manual will concentrate on MATLAB and this is likely the tool your demonstrators will know best. Try Octave if you wish, but do so "at your own risk".

1.1.3 Learning MATLAB

In order to become technically adept at using MATLAB it is important that you attempt all the exercises contained in this manual; as with most tools, MATLAB is best learned by actually *using* it to *do* mathematics, and this should be practised as often as possible. Try to incorporate MATLAB into your weekly problem sheets, by using it to check some of your hand-written answers. Examples include substituting your solution back into the equation and plotting an answer to make sure it makes sense.

In the rest of this chapter, we will give a brief introduction to MATLAB, outlining some of its basic features and illustrating them with some short examples. The output of each command line is not printed in this manual; after executing each line you should check that the output is as you would expect. If it is not, then you should think about why this might be, and if necessary consult your demonstrator.

1.2 The command prompt, variables, getting help

In MATLAB the prompt is the symbol ">>". Commands are entered to the right of the prompt, followed by the return/enter key. To illustrate this try entering the following:

```
>> 12/15
```

which should display 0.8000. Try some more commands

```
>> 3 + 18
>> 3+18
>> 3   + 18

>> 40*21
>> factorial(3)
```

1.2.1 Using MATLAB's built-in help system

There are various ways of getting help on MATLAB commands but one of them is built into the command prompt and is a good "first stop" for help. To use it, type `help` followed by the name of a command:

```
>> help factorial
```

You can also access help in various ways using the graphical user interface. (Don't worry if much of the information in the help text doesn't make sense yet.) Throughout this manual you should use the help facility to learn more about the commands you are introduced to.

1.2.2 Assigning to variables

Numbers can be assigned to variables

```
>> a = 4
>> b = factorial(4)
>> B = 5
```

Later you can recall the value of a variable

```
>> a
>> b
```

and variables can be used in further calculations

```
>> my_var = a^2 + b/10 + 2*B
```

It is important to understand what the symbol = means here: it is the *assignment operator*. B = 5 means "assign the value 5 to B". You could pronounce this as "B gets 5". You will sometimes see people writing the assignment operator as "$B \leftarrow 5$" and some other programming languages use "$B := 5$" to avoid confusion. For better or for worse, MATLAB uses =. Later on, we will meet the equality operator ==, which tests for equality and expresses symbolic equality: it can be pronounced "equals".

There are a few words that are reserved for MATLAB operations and so cannot be used for variable names. Examples include `while`, `for`, and `function`: we will encounter these later the course, and if you try to use one as variable name you'll usually get an error. But just because something is allowed doesn't mean it's a good idea:

```
>> pi = 3.2
>> sin(pi)
```

This seems like a good time to show you how to clear all variables.

```
>> clear
>> pi      % back to its usual value
>> a       % gives an error
```

Indeed, it is recommended that you start all exercises with `clear`. Let's say that again in bold:

> **Usually, you should start exercises with `clear`.**
> **When something doesn't work as expected, try `clear`.**

A semicolon suppresses the output. This is useful, for example, if the output is extensive and not specifically required. It can also make your code and output more readable; this will be more important as you start writing longer scripts in MATLAB.

```
>> a = 6;
>> b = 3;
>> c = a*b;
>> c = 2*(c + 6)
```

1.2.3 Floating-point numbers

By default MATLAB produces answers which are numerical approximations (albeit highly accurate approximations):

```
>> 5/3
>> pi
>> sqrt(2)
```

These numerical solutions are typically accurate to about 15 decimal places:

```
>> sin(0)
>> sin(pi)
```

Note the latter gives `1.2246e-16`, that is 1.2246×10^{-16}. You can increase the number of displayed digits with

```
>> format long
>> sin(pi)
```

but note this has no effect on the actual accuracy of the expression, just its display.

Internally, MATLAB by default stores numbers in "IEEE Double-Precision Floating Point Arithmetic" or just `double` for short. Each `double` takes 8 bytes (that is 64 bits, "binary digits") in the memory of the computer. The 64 bits is used to store numbers in scientific notation with a certain number of bits for the exponent (the "-16" in our example above) and the rest for the coefficient (1.222 in our example).

Computers are very fast at doing arithmetic on doubles, and this forms the basis for much of computation. For example, encoding/decoding your voice, music, and video on your phone, performing large-scale climate simulations, finding a location from GPS satellites, or finding eigenvalues of large matrices all involve billions of operators on doubles (or on similar structures of lower precision). A surprising number of things we do (e.g., searching the internet) are related to the linear algebra problem of "finding eigenvalues of large matrices".

However, sometimes it's nice when 6/9 becomes 2/3, not "0.666666666666667". Particularly when learning or studying mathematics, we are often less interested in the numerical solution of something but rather in the how and why of it. Think about the area under a curve (a number, rather easily approximated) versus the indefinite integral (a general expression). We want the computer to do both these tasks.

1.2.4 Symbolic computing

Symbolic manipulation software or computer algebra systems work with symbols rather than numbers. They try to work out derivatives and integrals and solve equations much the same way you do: by manipulating the symbols according to certain rules ("6/9: 6 is 2*3 and 9 is 3*3 so cancel the 3's to get 2/3").

The Symbolic Math Toolbox adds symbolic computing to MATLAB. We can check if we have the toolbox with the command:

```
>> ver
```

which might produce a lot of output but the important bit for us is

```
Symbolic Math Toolbox          Version 8.2     (R2018b)
```

This manual assumes that you are using Version 8.2 or any subsequent later versions of the Symbolic Math Toolbox. If you installed MATLAB according to the instructions at the beginning of this manual, this will certainly be the case.

With the toolbox, you define symbolic numbers at the MATLAB prompt:

```
>> sym('6/9')
>> sym('1')
>> b = sym('2')
>> c = sqrt(b)
>> c^2
```

Those are single quotes which surround a string. In fact, you can leave out the quotes when entering small integers or simple fractions.

```
>> sym(1)
>> sym(1000)
>> sym(6/9)
```

It is recommended to use this shortcut only for small integers and very simple fractions, and to revert to the quote form if there is any doubt. The following exercise should convince you there can indeed be doubt.

Exercise 1.1 Observe the output of `sym(13/10)`, `sym(133/100)`, `sym(1333/1000)` and follow the pattern a few more steps. What happens? Try the experiment again with quotes (`sym('13/10')`, etc). (If you're curious, `help sym` explains how the quoteless mechanism works.) □

Please keep in mind that the `sym` command expects to take as input an expression between quotes that represents a *number* or a *variable* (see Section 1.2.6). In particular, you shouldn't expect the `sym` command to handle more intrincate expressions that needs first to be evaluated; or MATLAB might complain. For example, try the following:

```
>> sym('6.0/9')
>> sym('2+5')
>> sym('exp(5)')
```

Starting from MATLAB release R2020a, you should use `sym(pi)` instead of `sym('pi')` to define a symbolic number equal to the value π. Although in previous releases the two commands could be used interchangeably, starting from release R2020a, `sym('pi')` defines an (unallocated) symbolic variable named `pi`, see Section 1.2.6.

1.2.5 Saving your work in `.m` file scripts

Instead of just typing commands into the command prompt, you can create a script. Within the graphical user interface, one way to do this is to select "New Script" from the menu bar. Add some commands to the file, for example:

```
% my solution to Exercise 1.1
sym(13/10)
sym(133/100)
```

Save the file as "ex1_1.m". From within the editor, you can run your script by clicking on the icon with the green triangle. You can also switch to the command prompt and type the name of your script (without the `.m`)

```
>> ex1_1
```

In either case, the contents of the script should execute in the command window. You can alternate between editing your script and running it.

It is recommended to save each exercise as a script, and you may want to use additional scripts as you work through this manual. Find what works for you, but remember, you may want to access this material later (for example, next term when you work through the projects).

1.2.6 Using symbols instead of numbers

Numbers are powerful, but the `sym()` command makes more interesting things possible because symbolic expressions can contain symbols:

```
>> x = sym('x')
>> y = asin(x^3)
>> z = sin(y)
```

Another example:

```
>> x = sym('x')
>> y = sym('y')
>> f = (x^3 + x^2) * sqrt(y) * exp(x)
```

In fact, there is a shortcut—`syms`—for defined symbolic variables:

```
>> syms x y z a b c r s t
>> f = sin(a*x) * sqrt(tan(r*t-x) + sin(t-1) * tanh(s^2*y))
```

Consider

```
>>    a = sym(6)
>>    b = sym(2)*a
>>    c = sym('2*a')
```

What happened? The result in b is correct, but c throws an error. As mentioned in the end of Section 1.2.4, this is because MATLAB does not allow you to write expressions inside quotes and pass them to sym. For example, instead of f = sym('a * x + b'), you should write

```
>>    syms a x b
>>    f = a*x + b
```

If you really need to pass a string instead of a number or a variable to sym, you can use the command str2sym. For example, you can write

```
>> str2sym('sqrt(2)')
```

However, passing expressions as strings may not always produce the desired result. For instance, compare the outputs of the following commands.

```
>>    a = sym(6)
>>    b = sym(2)*a
>>    c = str2sym('2*a')
```

In Section 2.1, we will learn about the subs command, which will help push assigned variables into expressions.

Variable types and casting

We have encountered two types ("classes") of objects so far: doubles, which store numbers with a finite precision, and syms, which store symbolic expressions. It will often be important to keep track of which is which. The whos command can help.

```
>> clear
>> a = sym('6/9')
>> b = 6/9
>> whos
```

Now we can ask some useful questions about what happens when you combine objects.

Exercise 1.2 Use MATLAB to evaluate the following:

```
>>    a = sym(2/3)
>>    b = 5/3
>>    c = a + b
>>    d = b*a
>>    e = a^b
>>    f = (2/3) ^ (sym(5/3))
>>    e - f
>>    whos
```

From these results, what does MATLAB typically do when it performs a binary operation which combines a double and a sym? This is known as casting. □

So very roughly speaking, as long as some symbols in your expression are of class sym, you can expect the result to be symbolic too. This is good news for entering complex expressions, for example:

```
>> x = sym('x');
>> f = 512*x^4 + 13/10*x^2 + 256*x^(2/3)
>> % because the alternative is just horrible:
>> g = sym(512)*x^(sym(4))  +  sym(13)/sym(10)*x^sym(2)  +  ...
>>    sym(256)*x^(sym(2)/sym(3))
>> f - g      % but they are the same in the end
```

The previous example also shows how to split a long line in two: simply interrupt the line with "..." (three full stops) and continue on the next. You'll also note that we have been using the character "%" to comment our code. This is a good idea to help the reader (often yourelf!) understand what the code does. As you've probably figured out, MATLAB ignores these comments when executing the code.

Sometimes it's hard to tell if an expression like f above was entered correctly; the output often looks as bad—or worse—than the input. Try this:

```
>> pretty(f)
```

That will render the expression in a way that you might find easier to decipher.[1] Similarly, `latex(f)` will output code that can be pasted into LATEX, which is the standard tool mathematicians use for typesetting mathematics.[2]

Just as `sym()` converts to symbolic expressions, `double()` converts to double-precision numerical values.

Exercise 1.3 Use MATLAB to evaluate the following exactly:

$$19 \times 99, \ 3^{20}, \ 2^{1000}, \ 25!, \ \frac{3}{13} - \frac{26}{27}.$$

Now calculate the decimal values of the following:

$$\frac{21}{23}, \ 17^{1/4}, \ \frac{1}{99!}, \ 0.216^{100}.$$

MATLAB has some names reserved for constants, some of which are shown in the following table.

pi	$3.1415\ldots$
i (or "1i")	$\sqrt{-1}$
inf	∞
nan (Not a Number)	$\frac{0}{0}, \sin(\infty)$

The number $e = 2.7182\ldots$, can be entered as

```
>> double_e = exp(1)
>> symbolic_e = exp(sym(1))
>> log(symbolic_e)
```

Exercise 1.4 *Careful:* it's not `sym(exp(1))`: try that and see what happens. Can you explain why?

As stated above, MATLAB denotes the square root of -1 by i (actually 1i or j work too). Suppose we wish to express $(2 + 3i)^4$ in the complex form $a + bi$, where a and b are real:

```
>> (2 + 3*i)^4
```

Some MATLAB programmers don't like this because they want to use i for a variable. Thus it is very common to instead write:

```
>> (2 + 3*1i)^4
>> % or
>> (2 + 3i)^4
```

These examples used class `double` but symbolic complex numbers work too:

```
>> syms x y
>> z = x + 1i*y
>> sin(z)
```

---

[1] Of course, beauty is subjective... The output of this command might have counted as "pretty" in 1992; perhaps "retro" might be a better word.

[2] Typesetting in LATEX is an invaluable skill to learn.

You might have a sense of unease at this point: you and I might be tacitly assuming x and y are real but of course MATLAB doesn't know that. Indeed the complex conjugate of $z = x + iy$ is $\bar{z} = x - iy$ but MATLAB says:

```
>> syms x y
>> z = x + 1i*y
>> conj(z)
```

We will see how to fix this later using assumptions in Section 2.2.2.

Exercise 1.5 Use MATLAB to express the following in the approximate decimal form $a + bi$:

$$(5 - 8i)^2, \quad i^{-1}, \quad i^2, \quad i^{\frac{1}{2}}, \quad \frac{1+i}{5+2i}, \quad \left(\frac{4}{3} + \frac{2i}{5}\right)^4.$$

□

Exercise 1.6 Use symbolic complex numbers to verify that $(a + bi)^2 = a^2 - b^2 + 2abi$. (Hint: one approach would be to construct the left-hand side and the right-hand side. Sometimes the `simplify()` command will give MATLAB a nudge...). □

1.2.7 Built-in functions and help

As we have seen already, MATLAB has many in-built functions, many of which are fairly obvious. The table below lists some of those that you should be familiar with.

`cos, cosh`	cosine and hyperbolic cosine
`sin, sinh`	sine and hyperbolic sine
`tan, tanh`	tangent and hyperbolic tangent
`sec, sech`	secant and hyperbolic secant
`csc, csch`	cosecant and hyperbolic cosecant
`cot, coth`	cotangent and hyperbolic cotangent
`exp`	exponential
`log`	natural logarithm
`log10`	base-10 logarithm
`abs`	absolute value (and magnitude of complex number)
`sqrt`	square root

MATLAB has extensive built-in help which you can access from the command prompt:

```
>> help cos
```

One important note is that help usually gives you the function that operates on `double` values. For example, with the command above, you'll probably see

```
Other functions named cos
    sym/cos
```

(you might have to click on the first line to see the second appear). If you are working with symbolic expressions, you can access the symbolic specific help:

```
>> help sym/cos
```

(An object-oriented programmer would say "cos is overloaded for `sym` input".) Other commands have associated help too:

```
>> help clear
>> help syms
>> help whos
```

You can also search for keywords:

```
>> lookfor hyperbolic
```

MATLAB has more detailed hyperlinked help accessed with the `doc` command (and also available online).

Exercise 1.7 Use MATLAB to evaluate the following:

$$\tan^2(\pi/3), \ \sec(\pi/6), \ 1 + \cot^2(\pi/4), \ \operatorname{cosec}^2(\pi/4)e^{3\ln 4}.$$

□

Exercise 1.8 Use the help facility to learn how to compute the binomial coefficient ^nC_r in MATLAB. Evaluate $^{10}C_4$ and $^{250}C_{12}$. Do this both symbolically and numerically. Which is the incorrect answer? □

Exercise 1.9 Evaluate $\arcsin(1/2)$ and $\arcsin((\sqrt{5}+1)/4)$. Evaluate $\sec(\arctan x)$ in terms of x. □

1.3 2D graphics

One of the very useful features of MATLAB is the ease with which many different types of graphs may be drawn. Most graphs are drawn with variants of the `plot` command, which has several forms, each with many optional arguments that determine the appearance of the resulting graph. Here we introduce a few of the simplest forms. If at any time you wish to learn more, explore with MATLAB's help and doc pages.

1.3.1 Simple line graphs

The simplest graphs are those of functions of a single real variable. For example, the graph of the function $\sin(10/(1+x^2))$ for $0 \le x \le 10$ is produced by the command

```
>> syms x
>> y = sin(10/(1+x^2));
>> fplot(y)
```

You should see the graph on your screen, and it should be blue, which is the default colour. You can control the domain with

```
>> fplot(y, [-5, 10])
```

The range of the vertical axis can also be specified:

```
>> fplot(y, [-5 10])
>> ylim([-1.2 1.2])
```

Exercise 1.10 Plot $\sin(10\cos x)$ on the interval $-2\pi \le x \le 2\pi$. □

You can draw more than one curve, in different colours and line styles. For example:[3]

---

[3]You will probably want to put longer examples like this in a script (see Section 1.2.5).

```
>> clear
>> syms x
>>
>> fplot(cos(x))
>> hold on
>> h = fplot(sin(x));
>> set(h, 'color', 'red')
>> set(h, 'linestyle', '--')
>> set(h, 'linewidth', 3)
>>
>> legend('cos', 'sin')
>> title('my colourful plots')
```

The use of "hold on" is crucial: try it without. The American spelling of "colour" is required. The variable h is called a "handle" and it gives you access to MATLAB "under the hood": try get(h) for example.

You can also clear a figure with clf and control which figure window you're drawing on:

```
>> figure(2), clf
>> h = fplot(atan(x));
>> set(h, 'linewidth', 4)
>> set(h, 'color', [0.5 0 0.5])
```

The colour was set using an "RGB triplet" a vector specifying the proportions of red, green and blue.

Exercise 1.11 Draw the graphs of these five functions on the same plot:

$$y_1(x) = 1, \quad y_2(x) = 1 + x, \quad y_3(x) = 1 + x + x^2/2, \quad y_4(x) = 1 + x + x^2/2 + x^3/6,$$

and $y_5(x) = \exp(x)$, on the interval $0 \le x \le 1$. Use different colours for each. Add a legend. □

You can interact with figures using the graphical user interface. For example, try zooming and panning with the mouse.

Exercise 1.12 Using a plot, find the approximate coordinates of all the real solutions of the nonlinear simultaneous equations

$$y = \sin x,$$
$$y = x^3 - 5x^2 + 4.$$

Hint: see help ginput for one interesting way to do this with a mouse. (You will see in Section 2.2.3 how to find numerical solutions to simultaneous equations such as these.) □

1.3.2   Parametric and polar plots

The command fplot can also plot a function that is defined parametrically (with x and y as functions of t, say). So the ellipse $x = 2\cos t$, $y = \sin t$ is plotted with the command

```
>> syms t
>> fplot(2*cos(t), sin(t), [0, 2*pi])
>> axis equal
```

The axis equal command will ensure that a circle looks like a circle.

Exercise 1.13 The cycloid $x = t - \sin t$, $y = 1 - \cos t$ is the curve traced out by a point on a wheel as the wheel turns. Plot this curve for $0 \le t \le 6\pi$. □

Polar plots of $r = f(\theta)$ for $a \le \theta \le b$ can be achieved with `ezpolar`[4], for example to draw the cardioid:

```
>> syms t
>> ezpolar(1 - cos(t), [0, 2*pi])
```

Exercise 1.14 Plot (both leaves of) the lemniscate $r^2 = \cos(2\theta)$. □

Alternatives to fplot

The `fplot` and `ezpolar` commands are useful to make quick plots, particularly of symbolic expressions. However, sometimes it might be useful to access the other MATLAB plotting commands in order to have more control over plots. We give one (non-exhaustive) example, as you will certainly encounter this style of MATLAB plotting:

```
>> syms x
>> y = sin(10/(1+x^2));
>> ym = matlabFunction(y)
>> xx = linspace(-5, 10, 512);
>> plot(xx, ym(xx))
```

The key idea here is to create a vector of `doubles` (the `linspace` command). We also convert the symbolic expression into a more traditional MATLAB function.

Animation

We finish this chapter with an animation:

```
>> clear
>> clf
>> syms x y
>> fsurf(real(atan(x + 1i*y)))
>> shading flat
>> camlight left
>> material shiny
>>
>> for a = 1:90
>>    view([-30, a])
>>    pause(0.1)
>> end
```

(More about `for` loops later.)

---

[4]The name is an abbreviation of 'easy polar': American pronunciation compulsory!

Chapter 2

Algebraic equations and calculus

In this chapter you will learn about:

- different ways of evaluating expressions,
- solving equations symbolically and numerically,
- using MATLAB to differentiate expressions,
- using MATLAB to evaluate integrals (symbolically and numerically),
- dealing with limits.

2.1 Evaluating expressions

We previously saw a small example of the behaviour of the MATLAB Symbolic Math Toolbox when redefining variables in an expression. Recall:

```
>> syms x
>> y = 2*x
>> x = sym(6)
>> y             % still 2x
```

The subs command is useful here.

```
>> subs(y)
```

By default subs substitutes the current value of all variables into the expression. In this case, x was 6 so it substitutes 6 in for x to give y = 12.

The subs command has a more specific form subs(expr, old, new) which replaces all occurrences of the expressions old in expr by the expression new. For example, suppose we want to evaluate the expression $x^2 + 3x - 2$ at $x = 1$. This is achieved by using the commands

```
>> syms x
>> y = x^2 + 3*x - 2;
>> subs(y, x, 1)
```

A useful feature is that y itself has not changed, so that if we now wish to evaluate y at a different value of x then this is easily done:

```
>> % first, note y unchanged
>> y
>> subs(y, x, 10)
>> subs(y, x, sym(pi))
>> y2 = subs(y, x, atan(x))
```

Exercise 2.1 In each of the following, evaluate the expression at the given value.

i) $x^3 - 3x^2 + 2x - 1$, at $x = 5$;

ii) $\sin x \cos^3 x$, at $x = \pi/4$;

iii) $\ln\left(u + 1 + \sqrt{u^2 - 3}\right)$, at $u = 2$.   □

A single call of `subs` can be used to perform several substitutions. As an example, suppose that we wish to evaluate $r = \sqrt{x^2 + y^2 + z^2}$ at the point $(x, y, z) = (1, 2, 3)$. This is achieved by typing

```
>> syms x y z
>> r = sqrt(x^2 + y^2 + z^2)
>> A = subs(r, [x y z], [1 2 3])
```

The `[]` notation indicates a vector: more on this later. Recall that you can use `double()` to evaluate to a decimal approximation:

```
>> double(A)
```

Sometimes after using `subs` you might need `simplify` to encourage MATLAB to clean up the result.

That completes this brief introduction to evaluation. We now move on to look at solving equations. The main aim is to introduce `solve()`, which can be used to solve equations, inequalities, and systems of these. First, recall that "=" is used in MATLAB to express assignment. To express equality we use "==".

```
>> a = sym(6)            % assign 6 to a
>> logical(a == 6)       % returns 1 for true
>> logical(a == 7)       % returns 0 for false
>> logical(a < 10)       % returns 1 for true
>> logical(a <= 6)       % returns 1 for true
>> logical(a ~= 5)       % returns 1 for true
>> logical(a ~= 6)       % returns 0 for false
```

These are mostly self-explanatory except for "~=" (tilde followed by equals) which means "not equal to".

2.2 The `solve` command

2.2.1 Solving equations symbolically

The `solve` command can be used to rearrange simple algebraic expressions to arrive at a new expression. For example, the solution of the equation

$$2x + 3 = 0 \tag{2.1}$$

for x can be obtained using the Symbolic Math Toolbox as follows:

```
>> syms x
>> eq = 2*x + 3 == 0
>> solve(eq, x)
```

Again please note the different uses of = and == here: `eq` is *assigned* the expression of Equation (2.1) which happens to contain an *equality* operator. Messing this up makes subtle bugs that can be hard to track down.

Note that the solution, $-\frac{3}{2}$, is not assigned to x. You can do that yourself if you wish:

```
>> x
>> x = solve(eq, x)
>> eq
>> eq2 = subs(eq)
```

Suppose that we wish to solve the simultaneous equations

$$x + y = 2, \qquad -x + 3y = 3.$$

This is done as follows:

```
>> clear
>> syms x y
>> eq1 = x+y == 2;
>> eq2 = -x+3*y == 3;
>> [xsoln, ysoln] = solve(eq1, eq2, x, y)
```

An alternative:

```
>> sol = solve(eq1, eq2, x, y)
>> xsoln = sol.x
>> ysoln = sol.y
```

where `sol` is something called a structure: roughly speaking it can contain "sub-variables" known as fields.

What about problems with multiple solutions?

```
>> clear
>> syms x
>> eqn = x^2 == -25
>> sol = solve(eqn, x)
>> sol(1)
>> sol(2)
```

Here `sol` is a vector of two solutions, $-5i$ and $5i$.

Exercise 2.2 Find the solutions of the following equations:

(a) $x^2 - x - 2025 = 0$, (b) $x^3 - 6x^2 - 19x + 24 = 0$,

(c) $2x^4 - 11x^3 - 20x^2 + 113x + 60 = 0$. ☐

Exercise 2.3 Use the `solve` command to find the point of intersection, in the (x, y)-plane, of the two lines

$$ax + by = A, \quad cx + dy = B.$$

Having found the intersection, find an expression for the distance of the point of intersection from the origin. ☐

Note that in the previous exercise the Symbolic Math Toolbox made various assumptions (for example that $ad - bc \neq 0$). Some other systems are very good at finding a wider variety of solution possibilities. This can be useful (and/or annoying) depending on what you're trying to accomplish.

2.2.2 Assumptions

Sometimes one makes assumptions about equations and variables. For example, you might be interested only in the real roots of a quadratic. Or you might have a mathematical model where you know a particular variable must stay positive (say it represents a positive physical quantity like concentration or density). The Symbolic Math Toolbox allows you to add assumptions to symbolic variables.

```
>> syms x
>> assume(x, 'real')
>> ineq = x^2 <= 25
>> S = solve(ineq, x, 'ReturnConditions', true);
>> S.conditions
>> assumeAlso(x > 0)
>> S = solve(ineq, x, 'ReturnConditions', true);
>> S.conditions
>> ineq = x^2 <= -25
>> S = solve(ineq, x, 'ReturnConditions', true);
>> S.conditions
```

To list the current assumptions, type:

```
>> assumptions
```

You can remove the assumptions on a particular variable using

```
>> x = sym('x', 'clear')
```

Recall that if you want to clear everything and start again, the usual approach is the `clear` command. But it is worth mentioning that this might not remove assumptions from variables.[1] The `clear all` command will do the right thing.

Exercise 2.4 Define $z = x + iy$ where x and y are symbolic variables. Take the complex conjugate of z. Now modify your code to assume that both x and y are real and run your code again. □

2.2.3 Solving equations numerically

The `solve` command is used for finding *symbolic* solutions to equations. This is not always possible, but *numerical* (approximate) solutions can usually be found. If `solve` is unable to find a symbolic solution, it will try to find a numerical solution.

For example, suppose we wish to solve the equation

$$\sin x = x^3 - 5x^2 + 4,$$

and that having plotted the graphs of $\sin x$ and $x^3 - 5x^2 + 4$ we know that there are three solutions, at approximately $x = -0.90, 0.89$ and 4.78 (these approximate solutions were obtained in Exercise 1.12).

To find these solution more precisely, we try `solve`:

```
>> clear
>> syms x
>> eqn = sin(x) == x^3 - 5*x^2 + 4;
>> solve(eqn, x)
```

This fails to find an exact symbolic solution but it finds a numerical solution

```
ans =
0.88543649189409090795806224578236
```

If you're going to *do something* else with the result, you probably want:

```
>> y = double(solve(eqn, x))
```

which converts it to a `double`.

What about the other two solutions? The most common numerical approaches involve providing either an interval in which to search or an initial guess. Here we outline two possible approaches.

---

[1]The variable might be deleted but not the assumption, possibly leading to very subtle and hard-to-find bugs. Determining whether the assumption will remain or not depends on the MATLAB release that you are using, and the way you redefine your new variables after having deleted them (the commands syms x or x = sym('x') might in this case behave differently).

The `vpasolve` command   The Symbolic Math Toolbox also has the command `vpasolve`, which uses "variable-precision arithmetic" (see Appendix B.2 for more information.) One way to use this command is to provide a search interval:

```
>> vpasolve(eqn, x, [-1 0])
>> vpasolve(eqn, x, [0 1])
>> vpasolve(eqn, x, [4 5])

ans =
-0.90040020886787653209849187387426
ans =
0.88543649189409090795806224578236
ans =
4.7813983927628242128187791632837
```

Alternatively, you can provide an initial guess:

```
>> vpasolve(eqn, x, -.9)
>> vpasolve(eqn, x, .8)
>> y = double(vpasolve(eqn, x, 4.7))
```

The last example includes conversion to `double`, assuming the solution is to be used in further calculations.

Note that the `vpasolve` command does not respect assumptions on variables.

Alternative approach: `fzero` for numerical root finding   The MATLAB command `fzero()` does root-finding: it searches for numerical solutions of $f(x) = 0$ where f is a function of one variable.

```
>> f = @(x)  sin(x) - ( x^3 - 5*x^2 + 4 );
>> format long
>> fzero(f, -1)

ans =
   -0.900400208867877
```

Some explanation: the first line creates a MATLAB "anonymous function" (see Appendix B.1), which is then passed to `fzero`, along with an initial guess. We can then find the third root:

```
>> fzero(f, 5)

ans =
   4.781398392762824
```

Note that `fzero` does not work with symbolic expressions or symbolic functions.

Having to enter the same expression again takes time and risks introducing an error. Here's one approach to converting the symbolic equation "eqn" from above into an appropriate function f:

```
>> % we want the left-hand-side and right-hand-side
>> LR = children(eqn)
>> f = matlabFunction(LR(1) - LR(2))
>> fzero(f, -1)

ans =
   -0.900400208867877
```

Exercise 2.5 Make a plot showing the roots of $x^3 + 3x^2 - 2x + 1$. Then use `solve()` to find the roots. Use `double()` to find numerical approximations to the roots. □

Exercise 2.6 Find real solutions of the equation

$$x \sin x = \frac{1}{2}$$

for x in the following ranges: (i) $0 < x < 2$; (ii) $2 < x < 4$; (iii) $6 < x < 7$; (iv) $8 < x < 10$. Verify graphically that there are no solutions in the range $4 < x < 6$.   □

2.3   Differentiation and the `diff` command

The Symbolic Math Toolbox can find the derivatives of expressions of one or several variables. In this section we introduce the `diff` command, which is the basic differentiation operator for expressions, and illustrate its use in different situations including finding the coordinates of stationary points.

We start with an example: the derivative of $\sin x$ with respect to x is found with the command

```
>> syms x
>> diff(sin(x), x)
```

The general syntax for differentiating an expression `expr` with respect to a variable `var` is

```
>> diff(expr, var);
```

and the derivative of $e^{x^2+a^2}$ with respect to x is found with the commands

```
>> syms x a
>> z = exp(x^2 + a^2);
>> d1 = diff(z, x)
```

The second derivative of $e^{x^2+a^2}$ with respect to x is

```
>> diff(d1, x)
```

and this same result can also be found all in one go by typing

```
>> diff(z, x, 2)
```

So the fifth derivative of $\sin(x^2)$ is achieved by

```
>> diff(sin(x^2), x, 5)
```

Partial differentiation[2] of expressions of more than one variable is performed by a straightforward extension of this procedure. For example, consider the expression $\sin x \cos y$. Its second derivatives can be found as follows. First define $z = \sin x \cos y$:

```
>> syms x y
>> z = sin(x)*cos(y)
```

Then the second derivative $\partial^2 z / \partial x^2$ is given by

```
>> diff(z, x, 2)
```

and $\partial^2 z / \partial y^2$ is found with

```
>> diff(z, y, 2)
```

or by nesting the `diff` command

```
>> diff(diff(z, y), y)
```

Thus one way to evaluate the mixed second derivative $\frac{\partial^2 z}{\partial x \partial y}$ is:

---

[2]This topic may be new to you; it will be covered in lectures this term.

```
>> diff(diff(z, x), y)
```

This is a bit cumbersome, particularly if you want something like $\frac{\partial^5 z}{\partial x^2 \partial y^3}$, and instead you can do:

```
>> diff(z, x, x, y, y, y)
```

For example, we can verify that the order of differentiation does not matter here:

```
>> diff(z, y, x)
>> diff(z, x, y)
```

Sometimes it is necessary to find the derivative of an expression at a particular point, rather than as a function of the independent variable(s). This is achieved with a combination of the `diff`, `subs` and sometimes `double` commands. Thus the value of the first derivative of $\sin x$ at $x = 6$ is given by the commands

```
>> f = diff(sin(x), x);
>> subs(f, x, 6)
```

and to get a decimal approximation:

```
>> double(subs(f, x, 6))
```

The following exercises give practice in the `diff` command and also revise some earlier commands such as `solve` and `plot`.

Exercise 2.7 Find
$$\frac{d}{dx} \left(\frac{(x^2 + 1)^4}{e^{2x}} \right)$$
and evaluate it at $x = 1$. ☐

Exercise 2.8 Plot the second derivative of $x^4 / (1 + x^2)$ for $-5 \le x \le 5$. ☐

Exercise 2.9 Given $y = 12x^5 - 15x^4 + 20x^3 - 330x^2 + 600x + 2$, find the (x, y) coordinates of any stationary points (that is, those for which $y'(x) = 0$). Use MATLAB to plot y so that the real stationary points are visible. Can you also make a plot to show the complex stationary points? ☐

Exercise 2.10 If $z = \ln(x^2 + y^2)$ find the value of the constant a such that

$$\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = ae^{-z}.$$

☐

Exercise 2.11 Execute this code:

```
>> diff(sym(5))
>> diff(5)
```

and explain the results (hint: what does `diff([1 2 5 9])` do?). ☐

2.3.1 Differentiation of unknown functions

So far only the derivatives of expressions with an explicit form (in terms of one or more independent variables) have been found. In this section we look at how to differentiate arbitrary functions, say $y(x)$ or $f(x, y)$, without first giving them an explicit form. For example, we might want to find dy/dx (as a function of y and x) given that $x^2 + y^2 = 3$. However, typing

```
>> syms x y
>> diff(y, x)
```

returns 0 because MATLAB does not know that we intended y to depend on x; x and y are free independent variables and are treated as such.

This difficulty is addressed in the Symbolic Math Toolbox by using y(x); MATLAB interprets this syntax to mean that y(x) depends on x. Thus we have

```
>> clear
>> syms y(x)
>> diff(y,x)
>> diff(y,x,2)
```

Note here the syms y(x) also makes x a symbolic variable.

Differentiation and the "symfun"

The syntax here is quite important. Note the difference

```
>> syms g(x)
>> A = diff(g, x)        % yes, A is a symfun for the deriv
>> B = diff(g(x), x)     % gives an expression, not a symfun
>> whos A B
```

The distinction between an expression and a symfun is usually not important, but in some cases—such as dealing with differential equations—it makes an important difference.

Functions of multiple variables can also be manipulated using symfuns:

```
>> syms f(x,y)
>> f
>> diff(f, x)
>> diff(f, x, 2)
>> diff(f, x, y)
>> diff(f, x, x, y, y, y)
```

Using symfuns, the diff command knows the usual rules of differentiation, for instance the addition, multiplication and quotient rules:

```
>> syms f(x) g(x)
>> diff(f + g, x)
>> diff(f * g, x)
>> diff(f / g, x)
```

To illustrate an application of this theory, suppose that we wish to find dy/dx (in terms of x and y) given that $x^2 + y^2 = 3$. To do this with MATLAB, first define the given explicit equation by typing

```
>> clear
>> syms y(x)
>> eq1 = x^2 + y(x)^2 == 3
```

and then differentiate this equation, assigning the output equation to a new name

```
>> eq2 = diff(eq1, x)
```

Let's ask MATLAB to solve the resulting expression for dy/dx:

```
>> solve(eq2, diff(y, x))          % fail
>> solve(eq2, diff(y(x), x))       % fail
```

Hmm, we'll have to try a bit harder:

```
>> % use subs to replace the deriv with "dydx"
>> syms dydx
>> eq3 = subs(eq2, diff(y(x), x), dydx)
>> % now solve for dydx
>> solve(eq3, dydx)
```

Exercise 2.12 Find the derivatives of

$$\sin f(x), \quad \sin\left(e^{f(x)}\right), \quad \exp\left(\sqrt{1 + f(x)g(x)}\right).$$

$\square$

Exercise 2.13 Find dz/dp implicitly (in terms of z and p) given that

$$p^3 + z(p)^2 + 3pz(p) = 0.$$

Try to convince MATLAB to isolate dz/dp, that is, solve for dz/dp. $\square$

2.4 Evaluating limits

This short section introduces the command `limit`. Its syntax is fairly self-explanatory. For example, to find

$$\lim_{x \to 0} \frac{\sin x}{x}$$

type

```
>> syms x
>> limit(sin(x)/x, x, 0)
```

Left and right limits can also be found. For example, to find

$$\lim_{x \to 3^+} \frac{x^2 - 4}{x^2 - 5x + 6}$$

use the command

```
>> syms x
>> expr = (x^2-4)/(x^2-5*x+6);
>> limit(expr, x, 3, 'right')
```

Exercise 2.14 Use MATLAB to evaluate the following limits

$$\text{(i) } \lim_{x \to 0} \frac{\tan x - x}{x - \sin x}, \quad \text{(ii) } \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

In each case, make a plot that clearly demonstrates that the limits are correct. $\square$

2.5   Integration

The Symbolic Math Toolbox can evaluate many integrals symbolically, and MATLAB itself can also provide numerical estimates of most definite integrals.

2.5.1   The `int` operator

The syntax for integrating an expression `expr` with respect to a variable `var` is `int(expr,var)`. For example, to integrate the expression xe^{ax^2} with respect to x type

```
>> syms x a
>> int(x*exp(a*x^2), x)
```

Note that the Symbolic Math Toolbox does not include the arbitrary constant of integration. To evaluate the definite integral

$$\int_0^1 xe^{5x^2}dx$$

type

```
>> int(x*exp(5*x^2), x, 0, 1)
```

and to evaluate

$$\int_0^u \frac{dx}{\sqrt{u-x}}$$

type

```
>> syms x u
>> int(1/sqrt(u-x), x, 0, u)
```

If the Symbolic Math Toolbox cannot evaluate an integral, it returns a representation of the integral itself:

```
>> int(x^x, x)
```

which is a correct answer but probably not what you were hoping for!

At this point you should experiment with some integrals just to see what can be done.

Exercise 2.15 Use MATLAB to integrate the following expressions with respect to x:

(i) $\sqrt{e^x - 1}$, (ii) $x^2(ax + b)^{5/2}$, (iii) $\sinh(6x)\sinh^4(x)$, (iv) $\cosh^{-6}(x)$, (v) $\sin(\ln(x))$,

where a and b are constants. ☐

Exercise 2.16 Use MATLAB to evaluate the following expressions symbolically:

(i) $\int_{1/2}^1 \frac{1}{1+x^3}dx$, (ii) $\int_0^1 x^2\tan^{-1}x\,dx$, (iii) $\int_0^\infty \frac{1}{(1+x)(1+x^2)}\,dx$. ☐

From these exercises you will note that the symbolic toolbox can deal with rather nasty integrals symbolically, and that it easily evaluates many of the integrals found in elementary texts or standard tables.

Nevertheless one often encounters integrals where it is desirable to have MATLAB evaluate an integral numerically—for example, because the Symbolic Math Toolbox fails to integrate or the symbolic calculation is too slow.

2.5.2 Quadrature: numerical evaluation of integrals

You may be aware that the majority of integrals cannot be evaluated symbolically. If the symbolic toolbox fails to find a symbolic answer to a definite integral, or if you know that the integral in question cannot be evaluated in terms of known functions, then it may be necessary to evaluate it numerically.

For example, suppose we try to integrate $\int_0^1 e^{-t}\arcsin(t)\,dt$, symbolically by typing

```
>> syms t
>> y = exp(-t)*asin(t);
>> z = int(y, t, 0, 1)
```

As we noted before, this returns the symbolic integral expression; the Symbolic Math Toolbox does not know the answer. MATLAB has a variety of commands for doing numerical integration. This is known as *quadrature*. These are independent of the symbolic toolbox, so we first convert out expression to a MATLAB function and call `integral()`. Continuing our example above:

```
>> ym = matlabFunction(y);
>> z = integral(ym, 0, 1)
```

The `integral` command has various accuracy options: see `help integral`.

To summarize, the `int()` command is part of the Symbolic Math Toolbox, and tries to evaluate integrals symbolically; the `integral()` command is a MATLAB command which evaluates integrals numerically using quadrature.

Exercise 2.17 Use MATLAB to evaluate the following integrals numerically to six-digit accuracy:

(i) $\displaystyle\int_0^1 e^{x^3}\,dx,$ (ii) $\displaystyle\int_0^{10} \frac{1}{\sqrt{1+x^4}}\,dx,$ (iii) $\displaystyle\int_0^5 \sin\left(e^{x/2}\right)\,dx.$ □

Exercise 2.18 In this exercise, we explore the function $\operatorname{sinc} x$, which is defined as

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

(a) Assign the expression `sin(x)/x` to a variable and then plot it for $-30 \le x \le 30$. (Note that fplot should have no problem with plotting $\frac{\sin 0}{0}$. However, you should satisfy yourself that the graph looks correct at $x = 0$, given the above definition.)

(b) First, it can be shown that each maximum or minimum of the graph of $\operatorname{sinc} x$ corresponds to a point of intersection of the graphs of $\operatorname{sinc} x$ and $\cos x$. Use MATLAB to illustrate this, by drawing the graphs of $\operatorname{sinc} x$ and $\cos x$ on the same plot for $0 \le x \le 10$. Then use `solve` to find numerical approximations for the x-coordinates of all stationary points of $\operatorname{sinc} x$ for $0 < x \le 10$, and verify that these are indeed where the two graphs that you have just drawn meet.

(c) Use MATLAB to evaluate the three integrals

(i) $\displaystyle\int_0^\infty \operatorname{sinc} x \, dx,$ (ii) $\displaystyle\int_0^\infty \operatorname{sinc} x \, \operatorname{sinc}\frac{x}{3} \, dx,$ (iii) $\displaystyle\int_0^\infty \operatorname{sinc} x \, \operatorname{sinc}\frac{x}{3} \, \operatorname{sinc}\frac{x}{5} \, dx.$

What do you think the value of $\displaystyle\int_0^\infty \prod_{k=0}^5 \operatorname{sinc}(x/(2k+1)) \, dx$ is? Use MATLAB to verify your conjecture.

□

Exercise 2.19 (Optional) In this exercise, we use MATLAB to investigate some rational bounds on π.

(a) Verify that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \tag{2.2}$$

This is in itself an interesting result, as $\frac{22}{7}$ is often used as a rational approximation for π. However, this result can also be used to obtain bounds on π in the form $a/b < \pi < c/d$, where a, b, c and d are integers.

(b) Verify by hand, without using MATLAB, that

$$\frac{1}{2}\int_0^1 x^4(1-x)^4 dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx < \int_0^1 x^4(1-x)^4 dx.$$

(c) Evaluate $J = \int_0^1 x^4(1-x)^4 dx$, and hence deduce that

$$\frac{1979}{630} < \pi < \frac{3959}{1260}. \tag{2.3}$$

This gives rational bounds on π. You can now use MATLAB to obtain tighter rational bounds, in the following manner. It can be shown that the identity (2.2) can be generalized by replacing the powers of 4 by powers of any integer multiple of 4 as follows.

$$\int_0^1 \frac{x^{4n}(1-x)^{4n}}{2^{2(n-1)}(1+x^2)}dx = (-1)^n(\pi - R_n),$$

where $n = 1, 2, 3, \ldots$ and R_n is a rational number. (In (2.2) above, $n = 1$ and $R_1 = 22/7$.) The Symbolic Math Toolbox should be able to verify this for any given n and compute the corresponding value of R_n. Then, following the procedure of parts (ii) and (iii) above, new bounds on π can be found.[3]

(d) Use MATLAB to calculate R_5.

(e) Hence find the upper and lower bounds on π for the case when $n = 5$. Evaluate these both as rational numbers and in decimal form. You might need variable precision arithmetic (Chapter B.2). □

---

[3]It can be shown that the higher the value of n, the tighter the bounds on π, but this is not considered here.

Chapter 3

Differential equations, sums, matrices and vectors

In this chapter you will learn about:

- solving differential equations,
- sums and products,
- the use of lists and vectors,
- 3D plotting.

3.1 Solving ordinary differential equations using `dsolve`

The "`dsolve`" command can be used to find the closed form solutions of many types of differential equations. For example, to solve the homogeneous differential equation

$$\frac{d^2y}{dx^2} - y = 0$$

we use the `symfun` class as follows:

```
>> syms y(x)
>> ode = diff(y,x,2) - y == 0
>> dsolve(ode)
```

Notice the way by which the symbolic toolbox gives the constants of integration. The important thing here is that, to obtain a `symfun` result for the command `dsolve`, `y(x)` was created as a `symfun` and `diff(y,x)` was used to create the ODE (not `diff(y(x),x)`). This is an example of the toolbox differentiating arbitrary functions, which was discussed in Section 2.3.1.

Inhomogeneous equations are no different. To solve

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

type

```
>> clear
>> syms y(x)
>> ode = diff(y,x) + y/x == x^2
>> dsolve(ode)
```

Sometimes MATLAB does not automatically simplify the solution as much as it could. For example, the solution of

$$\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = 16e^{3t}$$

is found by typing

```
>> syms x(t)
>> ode = diff(x,t,3) - 3*diff(x,t,2) + 3*diff(x,t) - x == 16*exp(3*t)
>> soln = dsolve(ode)
>> simplify(soln)
```

To insert initial and/or boundary conditions in MATLAB, proceed as in the following examples. The solution of the boundary value problem

$$y'' + 5y' + 6y = 0, \ y(0) = 0, \ y(1) = 3$$

is found as follows:

```
>> syms y(x)
>> DE = diff(y,x,2) + 5*diff(y,x) + 6*y == 0
>> soln = dsolve(DE, y(0)==0, y(1)==3)
```

We could then plot the solution:

```
>> fplot(soln, [0 1])
>> ylim([0 6])
>> grid on
>> ylabel('y')
```

Now consider the initial value problem

$$y'' = y, \ y(0) = 1, \ y'(0) = 0,$$

which could be solved as follows

```
>> syms y(t)
>> DE = diff(y,t,2) == y;
>> yp = diff(y,t);
>> Y = dsolve(DE, y(0)==1, yp(0)==0)
```

And again a plot:

```
>> Yp = diff(Y, t);
>> clf
>> fplot(Y, [0 2])
>> hold on
>> h = fplot(Yp, [0 2])
>> set(h, 'color', 'red', 'linestyle', '-.')
>> grid on
>> % note how we enter y'(x) here:
>> legend('y(x)', 'y''(x)')
```

Exercise 3.1 Use MATLAB to solve the following differential equations:

$$\text{(i)} \ \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = \sin x, \qquad \text{(ii)} \ \frac{dy}{dx} = 3xy, \ y(0) = 1.$$

In (ii), plot the solution. □

Exercise 3.2 Solve the differential equation $y'' + y = 0$, subject to the initial conditions $y(0) = a$, $y'(0) = b$. Assign the result to a variable and then evaluate the solution at $x = \pi$. □

Students interested in Chebfun (not part of this course) may note that it is a very powerful and flexible tool for solving differential equations numerically, both initial- and boundary-value problems. See chapters 7 and 10 of the Chebfun Guide at www.chebfun.org, and in particular, try the graphical user interface chebgui. Also check out Appendix B, "100 more examples", in the Chebfun-based text *Exploring ODEs*, freely available online at https://people.maths.ox.ac.uk/trefethen/ExplODE/.

3.2 Vectors and lists in MATLAB

MATLAB was originally designed to make it easy to manipulate matrices and vectors. A row vector takes the form [expr1, expr2, expr3,..., exprn], in other words it is a sequence of expressions enclosed in square brackets. We will often choose to think of this as a *list*. In a list, order and repetition are both important.

An example of entering a list is:

```
>> L = [1, 2, 3, 4]
```

In fact the commas are optional:

```
>> L = [10 12 11 10 9]
```

Specific elements can be extracted. Here we pick out the third element:

```
>> L(3)
```

In a similar fashion, you can replace entries of a list:

```
>> L(3) = 42;
>> L
```

There is a useful shortcut in list construction of the form "`start:step:end`", for example:

```
>> L1 = 1:10
>> L2 = 3:2:13
>> L3 = 17:-1:10
>> L4 = 1.0:0.2:1.6
```

Lists can be concatenated

```
>> clear
>> L1 = [1 2 3];
>> L2 = [4 5 6 7];
>> L3 = [L1 L2]
```

Note this makes a longer list rather than making a "list of lists". (The latter can be done in MATLAB using cell arrays: see Appendix B.3.) The number of elements in a list is its length:

```
>> n = length(L3)
```

You can make an empty list of length zero:

```
>> L = []
>> n = length(L)
```

3.2.1 Symbolic lists

We can also make lists/vectors of symbolic objects and manipulate their elements:

```
>> syms x
>> L = [sin(x) cos(x) tan(x)]
>> L(1)
>> diff(L(1), x)
>> % in fact we can differentiate all elements at once:
>> derivs = diff(L, x)
```

Every element of a list must be the same type. If you mix `sym` and `double` in a list construction, everything will be cast to `sym`:

```
>> clear
>> x = sym(10);   y = 11;   z = 12;
>> L = [x y z];
>> a = L(1);   b = L(2);   c = L(3);
>> whos
```

Notice that L is a 1x3 sym and the a, b, and c are all of class sym.

Exercise 3.3 Design an experiment to check what happens if you replace one element of an existing sym list with a double. On the other hand, what happens if you replace one element of an existing double list with a sym? □

3.2.2   Indexing in lists

Ranges of elements can be extracted using the colon notation:

```
>> L = 2:0.1:4
>> L(1:3)
>> L(2:2:12)
```

There is also a keyword "end":

```
>> L(1:end-4)
>> L(end-3:end)
```

You can use one list as indices to access another list:

```
>> I = [1 5 3 2]
>> L(I)
>> % or directly
>> L([1 5 3 2])
```

This leads to a common design pattern in MATLAB:

```
>> I = find(L > 3.05)
>> L(I)
>> I = find( (L >= 3.1) & (L < 3.8) )
>> L(I)
```

Here I is a list of indices satisfying the condition. You can combine the "list accessing a list" pattern with the "end" keyword. For example, the following will "rotate" a list:

```
>> rotLeft = L([2:end-1  1])
>> rotRight = L([end  2:end-1])
```

(This is a bit of a trick—so called "syntactic sugar"—e.g., you cannot store I = [2:end-1 1].)

3.2.3   Vector operations

To do component-wise operators on a vector, we need special "dot" operators. To raise each element to a power, use:

```
>> L = [1 2 4 8];
>> L3 = L .^ 3
```

Similarly, to multiple or divide the components of one vector by the components of a another vector (component-wise multiplication/division), do:

```
>> M = L .* L3
>> Z = (L.^4) ./ L - L3
```

The component-wise multiplication symbol is pronounced "dot star" or "dot times". The usual multiplication symbol * (without the dot) means matrix multiplication in this context (see Section 3.5).

Vectors and matrices are so fundamental in MATLAB that we return to them in Sections 3.5 and 3.6, but first we look at some applications of lists, namely for set theory and performing sums.

3.3 Sets

MATLAB doesn't have particular support for mathematical sets. (MuPAD does, using its worksheet mode.) However, using lists, we can implement some basic ideas of set theory.

The set $\{1,2,3\}$ could be represented by

```
>> S = [1 2 3]
```

Similarly the set of three polynomial expressions could be

```
>> syms x
>> S = [x^2 3*x 2]
```

Order should not be important and repetitions should be ignored in sets:

```
>> S = [1 3 4 1 2 1]   % not really a set
>> S = unique(S)       % much better
```

Given two sets A and B, MATLAB can perform the usual set operations of union $A \cup B$, intersection $A \cap B$ and difference $A \setminus B$:

```
>> A = [1 2 3 4]
>> B = [2 4 6 8]
>> union(A, B)
>> intersect(A, B)
>> setdiff(A, B)
```

Exercise 3.4 Use MATLAB to verify that if $X = \{1,2,3,5\}$, $Y = \{2,3,4\}$ and $Z = \{1,3\}$ then

$$X - (Y \cup Z) = (X - Y) \cap (X - Z) \ \text{ and } \ X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

(Of course this does not constitute a proof of the general results!) □

3.4 Sums and products

In this section we look at sums and products in both numerical and symbolic forms using the commands `sum`, `prod`, `symsum` and `symprod`.

3.4.1 Simple "numerical" sums

Suppose we want to evaluate the three sums

$$S_1 = \sum_{r=1}^{10} r, \qquad S_2 = \sum_{i=1}^{100} i^2, \qquad S_3 = \sum_{k=10}^{100} \frac{1}{(k+1)^2}.$$

One approach is to construct an appropriate vector and then add up the elements:

```
>> r = 1:10;
>> S1 = sum(r)
```

For S_2:

```
>> i = 1:100;
>> isqr = i.^2;
>> sum(isqr)
>> % or directly:
>> S2 = sum(i.^2)
```

Note that we must use the "dot" component-wise vector operations:

```
>> k = 10:100;
>> S3_num = sum(1 ./ (k+1).^2)
```

Exercise 3.5 Use MATLAB to evaluate the following:

$$\text{(i)} \ \sum_{r=1}^{10} (r^2 + 3r - 2), \qquad \text{(ii)} \ \sum_{i=1}^{50} 2^i, \qquad \text{(iii)} \ \sum_{k=1}^{10} e^{-\sqrt{k}}, \qquad \text{(iv)} \ \sum_{k=1}^{100} \frac{1}{\sqrt{k}}.$$

□

3.4.2 Products

Products can be approached in a similar fashion using the `prod` command. If you have not seen it before, $\prod$ denotes a product in the same way as $\sum$ denotes a sum. For example,

$$\prod_{r=1}^{5} (r+2) = 3 \times 4 \times 5 \times 6 \times 7 = 2520.$$

□

Exercise 3.6 Use MATLAB to evaluate the following:

$$\text{(i)} \ \prod_{r=1}^{5} (r+2), \qquad \text{(ii)} \ \prod_{r=1}^{10} r^2.$$

□

3.4.3 Symbolic manipulation of sums and products

The previous approach uses standard MATLAB (without the Symbolic Math Toolbox) and works for adding up numbers. But what about sums with variables in the limits? Or infinite sums? For this we use the `symsum` command.

Let us now consider a more general case, where the upper limit is replaced by a free variable, n. For example:

$$S_n = \sum_{r=1}^{n} r.$$

Our previous list idea won't work here, instead:

```
>> syms r n
>> Sn = symsum(r, r, 1, n)
```

(the first `r` here is the summand, the second is the variable to sum over, see `help symsum`). Now that we have S_n we can manipulate it in the usual ways:

```
>> subs(Sn, n, 10)
>> % which is the same as:
>> sum(1:10)
```

Infinite sums can be evaluated in this way too.

```
>> clear all
>> syms a k
>> S = symsum(a^k, k, 1, inf)
```

Why does this output look confusing? Recall that the geometric series converges only for certain values of a. Let's use the assumptions features of the symbolic toolbox to clear things up a bit:

```
>> syms a b
>> assume(b, 'real')
>> assumeAlso(b >= 0)
>> assumeAlso(b < 1)
>> assumptions(b)   % summarize assumptions on b
>> subs(S, a, b)    % replace a with b
```

Actually, that was "heavy handed", how about:

```
>> % this will clear the assumptions on b
>> b = sym('b', 'clear')
>> % want the magnitude of b to be less than 1
>> assume(abs(b) < 1)
>> assumptions(b)   % summarize assumptions on b
>> subs(S, a, b)    % replace a with b
```

Instead of using subs after the fact, you could also make the assumptions first:

```
>> clear all
>> syms a k
>> assume(abs(a) < 1)
>> S = symsum(a^k, k, 1, inf)
```

Exercise 3.7 Use MATLAB and the Symbolic Math Toolbox to evaluate the following:

$$\text{(i)} \sum_{k=0}^{n-1} a^k, \qquad \text{(ii)} \sum_{k=0}^{\infty} \frac{1}{k!}.$$

$\square$

Exercise 3.8 Use MATLAB and the Symbolic Math Toolbox to help verify the following:

$$\text{(i)} \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)} = 1, \qquad \text{(ii)} \sum_{k=0}^{\infty} \frac{k^2 + k - 1}{(k+2)!} = 0, \qquad \text{(iii)} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$\square$

Exercise 3.9 By embedding one symsum command within another, evaluate the double sums

$$\text{(i)} \sum_{m=0}^{10} \sum_{n=0}^{10} (2n+1)e^m, \qquad \text{(ii)} \sum_{n=0}^{10} \sum_{m=0}^{10} (2n+1)e^m, \qquad \text{(iii)} \sum_{m=0}^{10} \sum_{n=0}^{m} (2n+1)e^m,$$

where (i), (ii) should be approximately `4.2162e+06` and (iii) should be `3.814e+06`. $\square$

Exercise 3.10 Use symprod to evaluate the following:

$$\text{(i)} \prod_{r=1}^{10} r^2, \qquad \text{(ii)} \prod_{r=1}^{7} (1 - q^r), \qquad \text{(iii)} \prod_{r=1}^{\infty} q^r, \text{ where } q \in \mathbb{R}.$$

Check that substituting $q = 2$ gives `-78129765` in (ii). Does the Symbolic Math Toolbox give reasonable answers in (iii)? $\square$

3.5   Arrays, matrices and vectors

We saw horizontal row vectors in Section 3.2. Recall that the syntax was

```
>> h = [1 2 4 8]
```

The column vector $\mathbf{a} = [2,\ 1,\ 5,\ 9]^T$ can be defined in MATLAB by:

```
>> a = [2; 1; 5; 9]
```

As we saw in Section 3.2, to change the third element to the value 7, type

```
>> a(3) = 7;
```

and then check the result by typing

```
>> a
```

Vectors can be defined and initialized in various ways. For example a zero column vector can be set up with the command

```
>> c = zeros(4,1);
```

and then the entries may be input, for example, one at a time. Similar commands include `ones()` and `rand()`.

3.5.1   Matrix definition

Matrices—2D arrays of numbers—can be entered row by row:

```
>> A = [1 2 3; 4 5 6];
```

The entries of a matrix can be accessed by two indices

```
>> i = 1;
>> j = 1;
>> A(i,j)
```

or more directly as in

```
>> A(2,1)
```

Individual entries can be changed as necessary:

```
>> A(1,2) = -1;
>> A
```

MATLAB does not distinguish between matrices and vectors: they are all just *arrays* of various sizes. A MATLAB array can contain symbolic entries but all entries of an array must be of the same class.

3.5.2   Rows, columns and submatrices

Suppose that A is a matrix. We can extract the first row of A using

```
>> A = rand(4,5)
>> h = A(1, :)
```

and you can think of this as "first row, every column" (i.e., the first row of A). Similarly, we get the second column of A using:

```
>> v = A(:, 2)
```

We can also extract a submatrix

```
>> B = A(1:2, 1:3);
```

More generally, the submatrix of rows i to j and columns k to l can be extracted with the command `A(i:j, k:l)`.

3.5.3 Operators on matrices and vectors

As you will be aware, matrix algebra can only be performed on matrices of the right shapes, and we shall suppose that the operations defined below have a meaning. If the matrices are of the wrong shape, MATLAB will probably complain.

In the following table, A and B are matrices, $\mathbf{u}, \mathbf{v}$ are vectors and s a scalar.

MATLAB command	Function
`s*A`	Scalar multiplication by s
`A*B`	Matrix multiplication of A of B
`A*v`	Multiplication of A by vector v
`A+B`	Matrix addition of A and B (component-wise)
`A^n`	Matrix power
`A.^n`	Component-wise exponentiation
`A .* B`	Component-wise multiplication (A, B same size)
`A ./ B`	Component-wise division
`inv(A)`	Matrix inverse (square matrices)
`x = A \ b`	Solve systems of linear equations $Ax = b$
`A-s`	subtract scalar s from each element
`det(A), trace(A)`	determinant and trace of A
`transpose(A)`	transpose of A
`A'`	conjugate transpose of A
`u' * v`	inner product (dot product) of two *column* vectors
`u * v'`	outer product of two *column* vectors
`cross(u,v)`	cross product of two length-3 vectors
`eye(n)`	$n \times n$ identity matrix
`zeros(n,m)`	$n \times m$ zero matrix
`ones(n,m)`	$n \times m$ matrix of ones
`size(A)`	returns `[n m]` for an $n \times m$ matrix

Here are some examples:

```
>> u = rand(3,1)
>> v = rand(3,1)
>> u' * v
>> u * v'
```

The conjugate transpose and `transpose()` are the same thing for real matrices. MATLAB users tend to mostly use `'`.

In many cases, matrix operators work with symbolic matrices:

```
>> syms x
>> A = sym(round(10*rand(4,4)));
>> I = sym(eye(4,4));
>> B = A - x*I
>> inv(B)
>> pretty(inv(B))
```

The output of the above suggests that inverting matrices is rather hard. Happily, it's not something we have to do very often—for example, this is *not* how MATLAB's "\" solves $Ax = b$.

Exercise 3.11 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 7 \\ 4 & -5 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 4 & 6 \\ -1 & 0 \end{bmatrix}$.

Use MATLAB to find (where possible) $3A$, $A - 2B$, $A + 2C$, AC, CA, AB and A^T. Verify that $(AC)^T = C^T A^T$. □

3.5.4 Simultaneous equations

Exercise 3.12 Consider the set of simultaneous equations

$$3x + y - z = 1,$$
$$5x + y + 2z = 6,$$
$$4x - 2y - 3z = 3.$$

Write this as a system $A\mathbf{x} = \mathbf{b}$, where A is a 3×3 matrix, $\mathbf{x} = [x, y, z]^T$ and $\mathbf{b}$ is a column vector of length 3. Use the MATLAB backslash \ command to solve the system of simultaneous equations for x. □

3.5.5 Eigenvalues and eigenvectors

The eigenvalue problem for a square matrix A is to find nonzero eigenvectors x and associated eigenvalues λ such that:

$$Ax = \lambda x.$$

(If you haven't seen this before, you will soon in linear algebra—skip this for now and come back later). In MATLAB:

```
>> lambda = eig(A)
>> [V,D] = eig(A)
```

The first form gives a vector of the eigenvalues and the latter gives a diagonal matrix D with the eigenvalues on the diagonal. The columns of the matrix V contain the eigenvectors.

Exercise 3.13 (a) Use MATLAB to find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

(b) Now use MATLAB to find the eigenvalues and eigenvectors of

(i) A^3, (ii) A^{-1}, (iii) $A - 6I$ (where I is the identity), (iv) $(A + 3I)^{-1}$.

In each case, can you spot the relationship of the eigenvalues and eigenvectors found in (b) to those found in (a)? □

3.6 Meshgrids and 3D plots

We looked at indexing into vectors in Section 3.2.2. Understanding this enables concise code, particularly for plotting.

3.6.1 Logical masks

Let's reconsider an example from earlier using the `find` command:

```
>> L = 2:0.1:4;
>> % approach 1
>> I = find(L > 3.05);
>> L(I)
>> % approach 2
>> B = L > 3.05;
>> L(B)
```

The results of `L(I)` and `L(B)` are the same. But compare `I` and `B`. Note that `I` contains just the indices of those elements which are greater than 3.05. On the other hand, `B` contains 1's where the condition is true and 0's where it is false. `B` is known as a "logical mask".

Here's an example of what can be done with this sort of masking

```
>> x = linspace(-1, 1, 512);
>> y = cos(2*pi*x);
>> y(y > 0) = 0;
>> y(x > 0.5) = 2;
>> plot(x, y, 'b.-');
```

Exercise 3.14 Explain what each line of the previous job accomplishes. Change "512" in the above code to "20" and make the code explicitly display the two logical masks. If you swap the order of the two "y(...) = ..." lines, does the picture change? Why or why not? □

3.6.2 Meshgrid

We can build special 2D arrays (matrices) of x and y coordinates for use in plotting:

```
>> x = linspace(-2,2,60);
>> y = linspace(-1,1,40);
>> [xx,yy] = meshgrid(x,y);
>> f = sin(2*pi*xx) .* cos(pi*yy);
>>
>> figure(1), clf
>> surf(xx, yy, f);
>> axis equal
>> xlabel('x'), ylabel('y'), zlabel('f')
>>
>> figure(2), clf
>> pcolor(xx, yy, f);
>> axis equal, axis tight
>> xlabel('x'), ylabel('y')
>> colorbar
```

Meshgrid is essentially the Cartesian product of the 1D grids in the x and y directions. Examine the following output to understand what meshgrid does:

```
>> x = linspace(-2, 2, 5)
>> y = linspace(-1, 1, 4)
>> [xx, yy] = meshgrid(x, y)
```

3.6.3 3D plotting

Exercise 3.15 In the plotting code above, try adding "`shading interp`", "`shading flat`", or "`shading faceted`" after each plot.

After the `surf` command, try adding these command one after another and examining what each one does: "`shading interp`", "`camlight left`", "`lighting phong`", "`material shiny`".

Use your mouse and the rotate and zoom tools to interact with your figures. □

Next let's combine meshgrids with logical masks. This gives a way to modify regions for plotting.

```
>> x = linspace(-2,2,60);
>> y = linspace(-1,1,40);
>> [xx,yy] = meshgrid(x,y);
>> f = sin(2*pi*xx) .* cos(pi*yy);
>> I = (xx > 0)  &  (yy > 0);
>> f(I) = 0;    % or try: f(I) = f(I)+1;
>>
>> figure(1), clf
>> surf(xx, yy, f);
>> axis equal
>> xlabel('x'), ylabel('y'), zlabel('f')
>>
>> figure(2), clf
>> pcolor(xx, yy, f);
>> axis equal, axis tight
>> xlabel('x'), ylabel('y')
>> colorbar
```

Exercise 3.16 Modify the above code so that for every point where f is negative, the value is replaced with three times its absolute value.

Exercise 3.17 Modify the above code to plot $f = 3 - (x^2 + y^2)$ in the interior of the following triangle. Set $f = 0$ outside the triangle.
   The interior of the triangle is defined by the inequalities

$$y < 0.25x + 0.5, \quad x < 1.3 + 0.15y, \quad \text{and} \quad y > -0.4x - 0.5.$$

Also, try setting the outside values to `inf` and `nan` (not a number). □

Exercise 3.18 What does this code do?

```
>> A = rand(10,10);
>> I = A > 0.8;
>> A(~I) = 0;
```

□

Exercise 3.19 Plot the function `besselj(nu,r)` where $r = \sqrt{x^2 + y^2}$ on a circle of radius 20 for various integer values of ν. Animate your plots over $\nu = 0, 1, 2, \ldots$ using the `pause` command. □

Chapter 4

Loops, conditionals and functions

In this chapter you will learn about:

- using MATLAB loops and conditionals,
- writing functions in MATLAB,
- debugging programs.

In this chapter you will consolidate things you have learned so far and you will also be introduced to some simple programming techniques. The material here is very important and a lot of it will be needed in the Hilary Term projects; it is therefore crucial that you start Hilary Term Week 3 having completed Chapter 4.

In addition to being an environment for problem solving and exploring mathematics, MATLAB is also a programming language. By this we mean that a series of command lines can be written which, when executed, perform a particular task. You have seen several simple programs already (and hopefully written a few too). In this chapter you will look at programming in more detail, to develop longer and more structured command sequences. In particular you will look at some of the main building blocks of programming, namely loops, conditionals and functions.

4.1   Loops

Sometimes a MATLAB command (or a sequence of commands) may need to be executed a number of times. The `for...end` control structure is provided for this purpose. For example, say we want to output the squares of the integers from 1 to 5. This can be achieved with the following command:

```
>> for j = 1:5
>>    j^2
>> end
```

Essentially, any list can be provided in the "`for j = ...`" part, although you'll find most frequently it is some set of integers.

```
>> for i = 1:2:7
>>    i
>> end
>> for c = [4 7 1 4]
>>    x = sym(pi)/4*c
>>    s = sin(x)
>> end
```

As another example, a for loop could be used (instead of `sum`) to find the sum of the cubes of the even integers between 50 and 100 inclusive.

```
>> sumcubes  = 0;   % set sumcubes to zero initially
>> for i = 50:2:100
>>    sumcubes = sumcubes + i^3;
>> end
>> sumcubes
```

Note also that the command within the loop has been indented. This makes the whole structure easier for human beings to read (MATLAB doesn't care).

Exercise 4.1 Use a for loop structure to plot $\cos(n\pi x/2)$ for $n = 0, 1, 2, \ldots 10$. ☐

Exercise 4.2 Use one or more for loops to find $\sum_{n=1}^{N} (2n-1)^2$ for each $N = 5, 6, \ldots, 30$. ☐

Another form of loop is the "while" loop. For example, the sum-of-cubes-of-even-integers problem from above could be solved using a while loop as follows:

```
>> sumcubes = 0;
>> j = 50;
>>
>> while (j <= 100)
>>    sumcubes = sumcubes + j^3;
>>    j = j+2;
>> end
>> sumcubes
```

Exercise 4.3 In a while loop, is the condition—in the case above "(j<=100)"—checked *before* or *after* each iteration? ☐

4.1.1   Approximate solutions to equations

In this section the `for...end` structure is illustrated by looking at Newton's method for finding approximate solutions to equations of the form $f(x) = 0$. The process depends on the function being "approximately linear" when viewed on a small enough interval. (The First Year course should make this more precise.)

Starting with a guess at the solution, a, the equation of the tangent through $(a, f(a))$ is

$$\frac{y - f(a)}{x - a} = f'(a),$$

and this cuts the x-axis at $\hat{a} = a - \dfrac{f(a)}{f'(a)}$. This is (we hope) a better approximation to the root than a.

Say we want to find an approximate solution to

$$f(x) = x^5 - x^{1.33} - 1 = 0$$

and we know that an initial approximation to the solution we are interested in is $a = 1.5$ (e.g., from a plot). Newton's method can be used to find a better approximation with the following commands:

```
>> clear
>> syms x
>> f(x) = x^5 - x^(sym(133)/100) - 1
>> fp = diff(f)
>> % now use the correction in double
>> a = double(1.5);
>> a = a - double( f(a) / fp(a) )
```

If we repeat the last command over-and-over (try pressing the up-arrow), we ought to get a better-and-better approximation to the root. However, this is a little clumsy. Instead we use a loop, repeating until the approximation is accurate to some required number of significant figures:

```
>> % f and fp as above, then convert to matlab functions
>> f_m = matlabFunction(f)
>> fp_m = matlabFunction(fp)
>>
>> a = 1.5;
>> b = inf;
>> tol = 10^(-8);
>> while ( abs(b-a) > tol )
>>      b = a;
>>      a = a - f_m(a) / fp_m(a);
>> end
>> a
```

(Recall that double variables have roughly 15 decimal digits of accuracy, so convergence cannot be expected if `tol` is taken smaller than, say, `1e-14`.)

Exercise 4.4 Adapt the Newton's method example above to find a root of

$$f(x) = 5 - \frac{1}{4}\cos(3x) = x,$$

with a starting initial guess of $a = 5$. Show that the root is 5.24979 to six significant figures. How many iterations does it take? (Modify the code to tell you this.) Try to use `solve` to check your answer. □

4.2 Conditionals

The "if...else...end" construction allows us to choose alternative courses of action during the execution of a sequence of commands. The general structure is:

```
if <conditional expression>

  <statement sequence>

elseif <conditional expression>

  <statement sequence>

else

  <statement sequence>

end;
```

where there can be as many `elseif`'s as needed (or none at all) and `else` is also optional.

Here is a simple example:

```
>> x = -3
>> if (x > 2)
>>      y = x + 1;
>> elseif (x > 0) && (x <= 2)
>>      y = x;
>> else
>>      y = x - 1;
>> end
>> y
```

The $<conditional\ expression>$ mentioned above is a MATLAB statement which is either 0 (false) or non-zero (true). This will usually be a comparison using the equality and inequality operators on page 13 combined with logic operators: and ("&&", two ampersands), or ("||", two vertical bars), not ("~", tilde).

4.3 Functions

MATLAB functions are sets of commands which are needed to be used repeatedly, often with different parameter values each time. Some programming languages call these "procedures" to distinguish them from mathematical functions. At any rate, a MATLAB function could be used to define a (mathematical) function, to return a matrix, to plot a graph, or to perform a particular calculation. For example, create an file called "f2c.m" with the following contents:

```
function c = f2c(x)
%F2C  Convert Fahrenheit to Celsius
%   F2C(x) converts a temperature in degrees Fahrenheit to
%   degrees Celsius.

  c = 5/9*(x-32);

  % Typically, your function would be longer and would
  % include more steps here.

end   % optional
```

Having written this function, you can now use it on the MATLAB command prompt:

```
>> f2c(100)
```

to see that the answer is approximately 37.8°C. Once you have a function, it can be re-used easily:

```
>> f2c(0)
>> f2c(32)
>> f2c(80)
>> help f2c
>> lookfor celsius
```

The main points to notice are:

- Each function lives in its own .m file. The file name should match the function name.

- You can call functions which are either in the current working directory or in your path.

- Functions may have one or many arguments, or none at all.

- A function can return one or more values, or none at all.

- All communication between the function and whoever called it is through the arguments (inputs) and return values (outputs). That is, variables are "local" to the function: you cannot access variables from wherever the function was called from.[1]

One of the most important reasons to write functions is to encapsulate a smaller part of a problem: debug it, polish it, document it, and then use it as part of a larger program without worrying about how it works. Thus with a few well-designed functions, a programmer can avoid trying to think about her entire program at once and instead break it down in to manageable chunks.

To help a function be re-usable, there is a specifically formatted comment which forms the "help text" at the start of the function:

---

[1]Global variables are an exception. Most programmers avoid global variables whenever possible. However, type "help global" to find out more.

- The first line consists of the name of your function and a *short* summary:
  ```
  %function y = myfunc(x)
  %MYFUNC  What is does
  ```

- The remaining lines of the comment give detailed usage notes for your function:
  ```
  %function y = myfunc(x)
  %MYFUNC  What is does
  %   A more detailed explanation, by convention intended
  %   three spaces...
  ```

For shorter functions, there are also "inline functions" (also known as anonymous functions) which do not need a separate .m file. These can be created using the "@" syntax or they can be converted from symbolic expressions. More details can be found in Appendix B.1.

Exercise 4.5 Write a MATLAB function called c2f which converts temperatures in degrees Celsius to degrees Fahrenheit. Check your function by evaluating c2f(0) and c2f(37.8). □

Exercise 4.6 Write a function which defines the function f given by

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

Check your function by finding $f(2)$ and $f(-3)$. *Hint*: Use the if...end structure within your function. □

4.4   Examples of functions

This section presents some non-trivial examples of functions. In some cases, there may be existing implementations in MATLAB or we may not implement the "optimal" or most general approach. In particular, experienced MATLAB programmers will often try to solve problems by manipulating lists rather than using loops. We make no such attempt in this section. At any rate, if you think you know a better approach, implement it and try: the encapsulation principle of functions means it is simple to "drop in" a replacement implementation.

4.4.1   Finding the arithmetic mean of a set of numbers

This section revises the material on lists covered in Section 3.2. Suppose we wanted to find the arithmetic mean of a dataset given as a list of elements.

```
function s = arithmetic_mean(data)
%ARITHMETIC_MEAN  Find the arithmetic mean of a list of data

  n = length(data);
  s = 0;
  for i = 1:n
    s = s + data(i);
  end
  s = s / n;
end
```

An example call of this function would be:

```
>> arithmetic_mean([1 2 3 4 5 6])
```

4.4.2   Taylor's theorem

Taylor's theorem tells us how well a function which is sufficiently differentiable can be approximated by polynomials in the neighbourhood of a point. The `taylor` command in the Symbolic Math Toolbox calculates the Taylor series of a function together with an 'order' of the size of the remainder after the maximum degree polynomial terms which you specify. Thus

```
>> syms x
>> taylor(cos(x), x, 0, 'order', 5)
```

gives the Taylor expansion of $\cos(x)$ about zero with the remainder of degree five. The following function plots the function f and its Taylor polynomials about $x = a$ up to and including a maximal order for x in a specified interval.

```
function plot_taylor_poly(f, a, maxdeg, xlim)
%PLOT_TAYLOR_POLY Plot partial Taylor series of a function

  figure(1), clf
  h = fplot(f, xlim)
  set(h, 'linewidth', 2, 'linestyle', '--')
  leg{1} = char(f);
  hold on
  for i = 1:maxdeg
    tay = taylor(f, 'ExpansionPoint', a, 'Order', i)
    h = fplot(tay, xlim);
    set(h, 'color', [i/maxdeg 0 0])
    leg{i+1} = ['taylor' num2str(i-1)];
  end
  title('partial taylor series')
  legend(leg)
```

For example, try

```
>> syms u
>> f = cos(u)
>> plot_taylor_poly(f, 1, 4, [0 pi])
```

The command that builds the legend uses a cell-array (see Appendix B.3 for details).

4.4.3   Euler's method

You have seen how `dsolve` can find analytic solutions to some differential equations. However, not all differential equations have analytic solutions and in such cases approximate solutions may be computed using numerical methods. Matlab includes the commands `ode45` and `ode15s` for solving initial value problems, and as mentioned in section 3.1, Chebfun (not part of this course) is a convenient tool for both initial and boundary value problems. But here we write our own basic solver using Euler's method.

The Mean Value Theorem, which holds for a wide class of functions, says that $y(x + h) - y(x) = hy'(x + \theta h)$ for some θ depending on x and h. For suitable functions we can therefore hope that $hy'(x)$ is a reasonable approximation to $y(x + h) - y(x)$. This is the basis of "Euler's method".

For an equation in the form $y'(x) = f(x, y(x))$ together with an initial condition $y(a) = \alpha$, say, we first choose a "step length" h which gives discrete values of $x_i = a + (i - 1)h$, $i = 1, 2, \ldots$. Then set $y_1 = \alpha$ and compute approximations y_i to $y(x_i)$ for $i = 2, 3, \ldots$ using

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}).$$

Here is one possible implementation, `eulers_method.m`, of Euler's method for $y' = f(x, y)$, with $y(a) = \alpha$, for $a \le x \le b = a + nh$ (n steps of step size h). The x coordinates are output in `xx` and the solution sequence in `yy`.

```
function [xx,yy] = eulers_method(f, a, b, alpha, n)
%EULERS_METHOD  Solve initial value problem ODEs
%    [xx,yy] = eulers_method(f, a, b, alpha, n) calculates an
%    approximate solution to solves y'(x) = f(x,y(x)) with initial
%    condition y(a) = alpha up until x = b using n steps.
%
%    Increasing n generally increases accuracy.
%
%    This version works for scalar problems but can be generalized.

  h = (b - a) / n;    % stepsize

  % n steps means n+1 values, allocate storage for soln
  xx = linspace(a, b, n+1);
  yy = zeros(size(xx));

  % initial condition is first entry of solution
  yy(1) = alpha;

  % now loop, performing Euler updates
  for i = 2:(n+1)
    yy(i) = yy(i-1) + h*f(xx(i-1), yy(i-1));
  end
```

Hence the approximate solution to

$$\frac{dy}{dx} = f(x,y) = -xy, \qquad y(0) = 1,$$

for $0 \le x \le 10$ using 50 steps can be plotted as follows:

```
>> f = @(x,y)  (-x*y);
>> x0 = 0;  y0 = 1;
>> [x,y] = eulers_method(f, x0, 10, y0, 50);
>>
>> clf
>> plot(x0, y0, 'ro');
>> hold on;
>> plot(x, y, 'k.-');
```

For reference, here's how to use ode45 for this problem:

```
>> [xx2,yy2] = ode45(f, [0 10], y0);
>> plot(xx2,yy2, 'g.-')
>> legend('IC', 'Euler', 'ode45')
```

4.4.4 Euclid's algorithm

Euclid's algorithm for finding the greatest common divisor of two natural numbers is essentially based on the observation that if $a > b$ then $\gcd(a,b) = \gcd(a - b, b)$. Things can be improved by taking off as much b as possible:

```
function gcd = euclid(a,b)
%GCD    Find the greatest common denominator

  if b == 0
    gcd = a;
```

```
  else
    gcd = euclid(b, a - b * floor(a/b));
  end
```

Note this function is recursive—it calls itself.

4.4.5   A simple matrix function

Recall we looked at matrices in Section 3.5. The next function creates an $n \times n$ matrix with 0's on the diagonal, 1's everywhere below the diagonal and -1's everywhere above the diagonal.

```
function A = mymat(n)
%MYMAT  Make a matrix

  A = zeros(n,n);
  for i = 1:n
    for j = 1:n
      if (i > j)
        A(i,j) = 1;
      elseif (i < j)
        A(i,j) = -1;
      else
        A(i,j) = 0;
      end
    end
  end
```

This is perhaps not the most "stylish" way to write this function. A MATLAB pro might write a different function:

```
    function A = mymat2(n)
    %MYMAT  Make a matrix

      A = triu(-1*ones(n),1)  +  tril(ones(n),-1);
```

But they both work just fine:

```
    >> mymat(4)
    >> mymat2(4)
```

4.5   Debugging

When you have written a program, you should not be too surprised if it does not work the first time you try to run it. The program may either crash completely or may give the wrong answer. If you're really on a roll, you might take down MATLAB or your operating system too. Don't worry though, the rest of the internet is (probably) immune to the bugs in your MATLAB code (this should not be interpreted as a challenge).

- **If the program crashes:** This is the easy case, because you often know where to start. MATLAB's error message might point you to a line number. At any rate, you should try to locate the line where it dies.

  It may be helpful to remove semi-colons; the program will then print out intermediate calculations which should make it easier to find the offending line. You should then try to work out what is wrong. Remember that the error is very often *above* the line with the apparent error.

  Mixing up "=" and "==" is a very common syntax error. If you are copying code from a hardcopy, be careful about typing the letter O instead of the number zero (don't laugh: it happens).

- **If the program runs but gives the wrong answer:** This is the harder case; it may prove quite difficult to locate and correct the error. The first thing is to convince yourself that your program is trying to solve the right problem. You should try to run the program for very simple cases, and find the simplest case in which it gives the wrong answer. Remove all code that is not used in that particular calculation. This is relatively easy to do by inserting % to make them into comments.

  If you have written functions, test them separately to make sure they behave as you expect.

  Add some "debugging output" by removing semi-colons or otherwise printing out intermediate results. Trace this output manually to find a point where the results don't match what you expect.

- Watch out for **floating point** when you were expected **symbolic** results (and vice versa). "whos" is your friend.

- The MATLAB graphical user interface provides various features to help you debug code such as breakpoints. The command "keyboard" is useful for debugging functions: add this command to your function and when MATLAB gets there, it gives you access on the command line to the variables inside you function.

4.6 Further exercise

The following is a final optional exercise on functions.

Exercise 4.7 Write a function that displays the rows of Pascal's triangle up to and including the nth row for a given positive integer n. ☐

Appendix A

Simplification

A key strength of computer algebra systems like the Symbolic Math Toolbox is the ability to manipulate mathematical expressions symbolically. The symbolic manipulation of expressions occasionally (on a bad day you might say "often") creates results in a different form than what the user needs. It is thus useful to know how to persuade the toolbox to change the form of a particular expression into what is needed. This is more of an art than a science but experience and a working knowledge of the commands and techniques introduced here can be helpful.

A.1   Expand

The `expand` command does exactly what its name implies. In this section we will look at how it works when applied to polynomials and to trigonometric functions. For example the expansion of $(x+1)(x+z)^2$ is found by typing

```
>> syms x y z
>> expand((x+1)*(x+z)^2)
>> pretty(ans)
```

When applied to trigonometric functions `expand` uses the sum rules[1] to remove multiple angles, replacing them with powers:

```
>> expand(cos(2*x))
```

(remember that in the output `cos(x)^2` means $\cos^2 x$, etc.) Other examples of the `expand` command are

```
>> expand(sin(x+y))
>> expand(cos(2*x+y))
>> expand(cos(5*x))
>> pretty(ans)
```

Perhaps in the last example above we wished instead to obtain the partial expansion $\cos 5x = \cos 4x \cos x - \sin 4x \sin x$; we could use a little trickery:

```
>> c = expand(cos(x+y))
>> c2 = subs(c, y, 4*x)
```

Hyperbolic expressions can be expanded in much the same manner; for example try

```
>> expand(sinh(3*x))
```

---

[1] $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ is an example of a sum rule.

A.2 Factor

For polynomials the `factor` command is in some ways the opposite of `expand`, and often does as one would expect:

```
>> syms x y z
>> factor(x^5 - x^4 - 7*x^3 + x^2 + 6*x)
>> f = x^4 + 4*x^3*y - 7*x^2*y^2 - 22*x*y^3 + 24*y^4
>> factor(f)
```

In the second example above `f` has been defined first; this is a useful trick with long expressions to check that they are typed correctly before proceeding. The `factor` command can also be used on rational expressions:

```
>> g = factor( (x^3 - y^3) / (x^4 - y^4) )
>> pretty(g)
```

(where the common factor $(x - y)$ in the numerator and denominator has been cancelled).

Exercise 5.1 Expand the function $f = (x + y + 1)^3$ and then factorise $f - 1$. □

Exercise 5.2 Use MATLAB to show that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1),$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(You might want to review Section 3.4.3.) □

A.3 Collect

The command `collect` is useful when you want to collect terms in powers of a known expression.

```
>> syms x
>> g = (cos(x) + 1)*(cos(x) + 2)*(cos(x) + 3)
>> collect(g, cos(x))
```

A.4 Simplify

The command `simplify()` is the most general of the Symbolic Math Toolbox's simplification commands and is usually the one to be tried first. However, it is not always the most appropriate command and results can be unpredictable.

Positive integer powers of trigonometric and hyperbolic functions are simplified by this rule as far as possible, as illustrated by the following attempt to simplify $\sinh^4 x - \cosh^4 x$:

```
>> syms x y z
>> f = (sinh(x))^4 - (cosh(x))^4
>> simplify(f)
```

One of the most useful applications of `simplify` or `simple` is when trying to verify that `expr == expr2`, where `expr1` and `expr2` are both expressions. Here is example, an attempt to verify that

$$\frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \cosh 2x :$$

```
>> expr1 = (1+tanh(x)^2) / (1-tanh(x)^2);
>> expr2 = cosh(2*x);
>> simplify(expr1 - expr2)
```

The following exercises will give you some practice using the `simplify` command and also some earlier commands.

Exercise 5.3 Use `simplify` to show that

$$\text{if}\quad y = \frac{1 + \sin x}{1 + \cos x}\qquad\text{then}\qquad \frac{dy}{dx} = \frac{\cos x + \sin x + 1}{1 + 2\cos x + \cos^2 x};$$

$$\text{if}\quad y = \ln\sqrt{\frac{1 + x}{1 - x}}\qquad\text{then}\qquad \frac{dy}{dx} = -\frac{1}{-1 + x^2};$$

$$\text{if}\quad y = \ln\left(\frac{(1 + x)^{1/2}}{(1 - x)^{1/3}}\right)\qquad\text{then}\qquad \frac{dy}{dx} = \frac{-5 + x}{6(-1 + x^2)}.$$

□

Exercise 5.4 Consider the general cubic polynomial

$$f(x) = \frac{1}{3}ax^3 + bx^2 + cx + d,$$

where a, b, c and d are real constants. If the stationary points of f are at x_1 and x_2, use MATLAB to show that

$$f(x_1) - f(x_2) = -\frac{a}{6}(x_1 - x_2)^3.$$

□

Exercise 5.5 Show that the function $f = 1/r$, where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$ and a, b and c are constants, is a solution of Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

□

Appendix B

Other advanced topics

B.1   MATLAB "handle" functions and anonymous functions

We have seen that functions can be created using `.m` files (Section 4.3). You can create function handles using "anonymous functions". This is most helpful for making a short function, for example:

```
>> alpha = 0.5
>> y = @(x) sin(alpha*x)
>> y(pi)
```

You can also create these sorts of functions from symbolic expressions:

```
>> syms x
>> y = sin(x)
>> ym = matlabFunction(y)
```

See `help function_handle` and `help matlabFunction` for more information.

B.2   Variable precision arithmetic

"Out-of-the-box" MATLAB focuses on supporting double precision floating point (the `double` class). Recall this class represents numbers using a precision of roughly 15 decimal digits. The Symbolic Math Toolbox (based on MuPAD) adds support for symbolic expressions (using the `sym` and `symfun` classes). The Symbolic Math Toolbox also adds support for Variable Precision Arithmetic via the `vpa` class. Having floating point numbers with more than 15 digits can be a useful feature and so this chapter is essentially for your own interest or reference.

To use variable precision arithmetic, first set the desired precision. Then use `vpa()` or a quoted symbolic constructor with a decimal place.

```
>> digits(64)
>> a = vpa(2/3)
>> b = sym('2.0')/3
```

Note that `b = sym(2.0)/3` won't work (why not?). Probably the `vpa()` command is less prone to accidental unwanted use.

It is worth noting that the `clear` command does not reset digits to its default value. But `clear all` will.

Exercise 6.1 All rational numbers p/q, where p and q are integers ($q \neq 0$), have a terminating or (eventually) repeating decimal form. By increasing the precision using `digits()` as appropriate, find the exact decimal form of 21/23. □

Exercise 6.2 The following two expressions are both approximations to π that were discovered by the Indian mathematician Ramanujan (1887–1920):

$$\pi_1 = \frac{12}{\sqrt{190}} \ln\left((2\sqrt{2} + \sqrt{10})(3 + \sqrt{10})\right)$$

and

$$\pi_2 = \sqrt{\sqrt{9^2 + \frac{19^2}{22}}}.$$

Use VPA to find the absolute errors $|\pi_1 - \pi|$ and $|\pi_2 - \pi|$. Hence determine how good these approximations actually are. □

B.3 Cell Arrays

Recall that a vector in MATLAB is homogeneous: everything inside it must be the same type (`double`, `sym`, etc). How dull! Cell arrays on the other hand can store anything, even other cell arrays. Make one like this:

```
>> A = {'Freedom!', sym(2/3), rand(3,3)}
>> A{1}
>> A{2}
>> A{3}
```

A common use of cell arrays is to store a "list of strings". This is because an array of strings just concatenates them. For example:

```
>> L = ['red' 'lorry' 'yellow' 'lorry']
>> % output is 'redlorryyellowlorry'
>> S = {'red' 'lorry' 'yellow' 'lorry'}
>> % output is a 4-element cell array
```

This is useful for constructing legends for plots when using loops.

Academic Honesty

In Michaelmas Term you are encouraged to co-operate with fellow students in learning to use the system and to use MATLAB. The Hilary Term projects, however, must be your own unaided work.

The description of each project gives references to books covering the relevant mathematics. If you cannot understand some of this then you are free to consult your tutors or others about it, but not about the project itself. You may discuss with the demonstrators and others the techniques described in the Michaelmas Term Student Guide, and the commands listed in the Hilary Term Student Guide or found in the MATLAB Help pages. You may ask the course director to clarify any obscure points in the projects.

The projects must be your own unaided work. You will be asked to make a declaration to that effect when you submit them.

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