C4.3 Functional Analytic Methods for PDEs
Lecture 3

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In the last lecture

- Duals of Lebesgue spaces.
- L^2 as a Hilbert space.
- Density of simple functions for Lebesgue spaces.

This lecture

- Weak and weak* convergence in Lebesgue spaces.
- Continuity property of translation operators in L^p.
- Convolution. Young's inequality.
- Differentiation rule for convolution.

Weak and weak* convergence in L^p

Definition

Let X be a normed vector space and X^* its dual.

1. (i) We say that a sequence (x_n) in X converges weakly to some $x \in X$ if $Tx_n \to Tx$ for all $T \in X^*$. We write $x_n \rightharpoonup x$.
2. (ii) We say that a sequence (T_n) in X^* converges weakly* to some $T \in X^*$ if $T_n x \to Tx$ for all $x \in X$. We write $T_n \rightharpoonup^* T$.

Weak sequential compactness

Theorem (Weak sequential compactness in reflexive Banach spaces)

Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

Corollary

Assume that $1 < p < \infty$ *and* (f_k) *is bounded in* $L^p(E)$*. Then there is a subsequence* f_{k_j} *which converges weakly in* L^p*. In other words, there exists a function* $f \in L^p$ *such that*

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^{p'}(E).$$

Weak* sequential compactness

Theorem (Helly's theorem on weak* sequential compactness in duals of separable Banach spaces)

Every bounded sequence in the dual of a separable Banach space has a weakly convergent subsequence.*

Corollary

Assume that (f_k) *is bounded in* $L^\infty(E)$*. Then there is a subsequence* f_{k_j} *which converges weakly* in* L^∞*. In other words, there exists a function* $f \in L^\infty$ *such that*

$$\int_E f_{k_j} g \to \int_E fg \text{ for all } g \in L^1(E).$$

A summary

	Dual	Reflexivity	Separability	Sequential compactness of $\overline{B(0,1)}$
L^p $1 < p < \infty$	$L^{p'}$	Yes	Yes	Weak and weak*
L^1	L^∞	No	Yes	Neither
L^∞	$\supsetneq L^1$	No	No	Weak*

Continuity of translation operators

Translation operators: For a $h \in \mathbb{R}^n$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, define $\tau_h f$ by

$$(\tau_h f)(x) = f(x+h) \text{ for all } x \in \mathbb{R}^n.$$

Then $\tau_h : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear transformation for $1 \leq p \leq \infty$. In fact it is an isometric isomorphism.

Theorem (Continuity in L^p)

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then

$$\lim_{|h| \to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Continuity of translation operators

- In other words, for $1 \leq p < \infty$, for every fixed $f \in L^p(\mathbb{R}^n)$, the map $h \mapsto \tau_h f$ is a continuous map from $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$.
- The theorem is false for $p = \infty$, e.g. with $f = \chi_Q$ with Q being the unit cube.
- The theorem does ***NOT*** assert that the maps $h \mapsto \tau_h$ is a continuous map from $\mathbb{R}^n$ into $\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$. In fact,

$$\|\tau_h - Id\|_{\mathscr{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))} \geq 2^{1/p} \text{ when } h \neq 0.$$

  - ⋆ Let $r = |h|/4$ and let $f = c_n r^{-n/p} \chi_{B_r(0)}$ where c_n is chosen such that $\|f\|_{L^p} = 1$.
  - ⋆ Then $\tau_h f$ and f has disjoint support. So

$$\|\tau_h f - f\|_{L^p} = \left\{\|\tau_h f\|_{L^p}^p + \|f\|_{L^p}^p\right\}^{1/p} = 2^{1/p}.$$

Continuity of translation operators

Proof:

- Let $\mathscr{A}$ denote the set of functions f in L^p such that $\|\tau_h f - f\|_{L^p} \to 0$ as $|h| \to 0$.
- It is clear that if $f, g \in \mathscr{A}$ then $f + g \in \mathscr{A}$, and $\lambda f \in \mathscr{A}$ for any $\lambda \in \mathbb{R}$. So $\mathscr{A}$ is a vector subspace of L^p.
- We claim that $\mathscr{A}$ is closed in L^p, i.e. if $(f_k) \subset \mathscr{A}$ and $f_k \to f$ in L^p, then $f \in \mathscr{A}$. Indeed, by Minkowski's inequality, we have

$$\begin{aligned}\|\tau_h f - f\|_{L^p} &\leq \|\tau_h f_k - f_k\|_{L^p} + \|\tau_h f_k - \tau_h f\|_{L^p} + \|f_k - f\|_{L^p}\\ &= \|\tau_h f_k - f_k\|_{L^p} + 2\|f_k - f\|_{L^p}.\end{aligned}$$

  Now, if one is given an $\varepsilon > 0$, one can first select large k such that $\|f_k - f\|_{L^p} \leq \varepsilon/3$, and then select $\delta > 0$ such that $\|\tau_h f_k - f_k\|_{L^p} \leq \varepsilon/3$ for all $|h| \leq \delta$, so that

$$\|\tau_h f - f\|_{L^p} \leq \varepsilon \text{ for all } |h| \leq \delta.$$

Continuity of translation operators

- So $\mathscr{A}$ is a closed vector subspace of L^p.
- Now, observe that if Q is a cube in $\mathbb{R}^n$, then $\|\tau_h \chi_Q - \chi_Q\|_{L^p} \rightarrow 0$ as $|h| \rightarrow 0$, by e.g. Lebesgue's dominated convergence theorem (or a direct estimate).
- So $\mathscr{A}$ contains all finite linear combinations of characteristic functions of cubes. In particular, it contains all finite rational linear combinations of characteristic functions of cubes belonging to a fixed class of dyadic cubes. As this latter set is dense in L^p and $\mathscr{A}$ is closed, we thus have $\mathscr{A} = L^p$, as desired.

Convolution

Definition

Let f and g be measurable functions on $\mathbb{R}^n$. The convolution $f * g$ of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\,dy$$

wherever the integral converges.

Young's convolution inequality

Theorem (Young's convolution inequality)

Let p, q and r satisfy $1 \le p, q, r \le \infty$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Young's convolution inequality

Proof: We will only deal with the case $q = 1$ and $r = p$. We are thus given $f \in L^p, g \in L^1$. We need to show that $f * g \in L^p$ and $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.

- Observe that $|f * g| \leq |f| * |g|$. We may thus assume without loss of generality in the proof that $f, g \geq 0$.
- Case 1: $p = 1$.
  - ⋆ Consider the integral

$$I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y) g(x - y)\, dx\, dy.$$

    This integral is well-defined as $f, g \geq 0$ and the function $G(x, y) = g(x - y)$ is measurable as a function from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$.

Young's convolution inequality

- Case 1: $p = 1$.
  - ⋆ Consider $I = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(y)g(x-y)\,dx\,dy$.
  - ⋆ By Tonelli's theorem, we have

$$I = \int_{\mathbb{R}^n} \Big\{ \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy \Big\}\,dx = \int_{\mathbb{R}^n} (f * g)(x)\,dx$$
$$= \|f * g\|_{L^1}.$$
$$I = \int_{\mathbb{R}^n} f(y) \Big\{ \int_{\mathbb{R}^n} g(x-y)\,dx \Big\}\,dy = \int_{\mathbb{R}^n} f(y)\|g\|_{L^1}\,dy$$
$$= \|f\|_{L^1}\|g\|_{L^1}.$$

  - ⋆ So $\|f * g\|_{L^1} = \|f\|_{L^1}\|g\|_{L^1}$.

Young's convolution inequality

- Case 2: $p = \infty$. This case is easy, as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$
$$\leq \int_{\mathbb{R}^n} \|f\|_{L^\infty}\, g(x-y)\, dy = \|f\|_{L^\infty}\, \|g\|_{L^1}.$$

- Case 3: $1 < p < \infty$.
  - ⋆ We start by writing

$$|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y) g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\, dy$$

and applying Hölder's inequality to the above.

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - ⋆ $|(f * g)(x)| = \int_{\mathbb{R}^n} [f(y) g(x-y)^{\frac{1}{p}}][g(x-y)^{\frac{1}{p'}}]\, dy$.
  - ⋆ So

$$\begin{aligned} |(f * g)(x)| &\leq \Big\{ \int_{\mathbb{R}^n} f(y)^p g(x-y)]\, dy \Big\}^{1/p} \Big\{ \int_{\mathbb{R}^n} g(x-y)\, dy \Big\}^{1/p'} \\ &= [(f^p * g)(x)]^{1/p} \|g\|_{L^1}^{1/p'}. \end{aligned}$$

  - ⋆ It follows that

$$\begin{aligned} \|f * g\|_{L^p} &= \Big\{ \int_{\mathbb{R}^n} |(f * g)(x)|^p\, dx \Big\}^{1/p} \\ &\leq \Big\{ \int_{\mathbb{R}^n} (f^p * g)(x)\, dx \Big\}^{1/p} \|g\|_{L^1}^{1/p'} \\ &= \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'} \end{aligned}$$

Young's convolution inequality

- Case 3: $1 < p < \infty$.
  - ⋆ $\|f * g\|_{L^p} \leq \|f^p * g\|_{L^1}^{1/p} \|g\|_{L^1}^{1/p'}$.
  - ⋆ So by Case 1,

$$\|f * g\|_{L^p} \leq \Big[\|f^p\|_{L^1} \|g\|_{L^1}\Big]^{1/p} \|g\|_{L^1}^{1/p'}$$
$$= \|f\|_{L^p} \|g\|_{L^1}.$$

Some notations

- If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
- If f is a function and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, we write $\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f$.
- For $k \geq 0$, $C^k(\mathbb{R}^n) = \Big\{$continuous $f : \mathbb{R}^n \to \mathbb{R}$ such that $\partial^\alpha f$ exists and is continuous whenever$|\alpha| \leq k\Big\}$.
- $C_c^k(\mathbb{R}^n) = \Big\{f \in C^k(\mathbb{R}^n)$ which has compact support$\Big\}$. Recall that, for a continuous function f,

$$Supp(f) = \text{Support of } f = \overline{\{f(x) \neq 0\}}.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in C_c^0(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.

Proof:

- Fix some $x \in \mathbb{R}^n$. We need to show that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$.
- We compute

$$\begin{aligned} &f * g(x+z) - f * g(x) \\ &\quad = \int_{\mathbb{R}^n} f(y) g(x+z-y)\,dy - \int_{\mathbb{R}^n} f(y) g(x-y)\,dy \\ &\quad = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\,dy. \end{aligned}$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- $f * g(x+z) - f * g(x) = \int_{\mathbb{R}^n} f(y)[g(x+z-y) - g(x-y)]\,dy$.
- Since $g \in C_c^0(\mathbb{R}^n)$, $g \equiv 0$ outside of some big ball B_R centered at 0. Then, for $|z| < R$,

$$f * g(x+z) - f * g(x) = \int_{|x-y| \leq 2R} f(y)[g(x+z-y) - g(x-y)]\,dy.$$

- Note that as g is continuous, it is uniformly continuous on $\bar{B}_{3R}$. Thus, for any given $\varepsilon > 0$, there exists small $\delta \in (0, R)$ such that

$$|g(x+z-y) - g(x-y)| \leq \varepsilon$$

whenever $|z| \leq \delta$ and $|x-y| \leq 2R$.

- So when $|z| \leq \delta$, we have

$$|f * g(x+z) - f * g(x)| \leq \varepsilon \int_{|x-y| \leq 2R} |f(y)|\,dy.$$

Convolution with a function in $C_c^0(\mathbb{R}^n)$

Proof:

- So when $|z| \leq \delta$, we have

$$\begin{aligned} |f * g(x+z) - f * g(x)| &\leq \varepsilon \|f\|_{L^1(\{|x-y| \leq 2R\})} \\ &\leq \varepsilon \|f\|_{L^p(\mathbb{R}^n)} \|1\|_{L^{p'}(\{|x-y| \leq 2R\})} \\ &= C_n R^{n/p'} \|f\|_{L^p} \varepsilon. \end{aligned}$$

- Since the right side can be made arbitrarily small, this precisely means that $f * g(x+z) - f * g(x) \to 0$ as $z \to 0$, i.e. $f * g$ is continuous.

Differentiation rule for convolution

Lemma

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, *and* $g \in C_c^k(\mathbb{R}^n)$ *for some* $k \ge 1$, *then* $f * g \in C^k(\mathbb{R}^n)$ *and*

$$D^\alpha (f * g)(x) = (f * D^\alpha g)(x) \text{ for all multi-index } \alpha \text{ with } |\alpha| \le k.$$

Proof

- We will only consider the case $k = 1$. The general case can be proved by applying the case $k = 1$ repeatedly.
- Suppose that $g \in C_c^1(\mathbb{R}^n)$. Fix a point x and consider $\partial_{x_1}(f * g)(x)$. We need to show that

$$\lim_{t \to 0} \underbrace{\frac{(f * g)(x + te_1) - f * g(x)}{t}}_{=: D.Q.(x,t)} = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- We have

$$D.Q.(x,t) = \int_{\mathbb{R}^n} f(y) \frac{g(x - y + te_1) - g(x - y)}{t}\, dy.$$

As $t \to 0$, the integrand converges to $f(y)\partial_{x_1} g(x - y)$. We would like to show that the above integral converges to

$$\int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x - y)\, dy = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- As before, if the support of g is contained in B_R, then, for $|t| < R$,

$$D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y)\frac{g(x-y+te_1) - g(x-y)}{t}\,dy.$$

- When $|x-y| \leq 2R$ and $|t| < R$, we have $|x-y+te_1| \leq 3R$. Hence

$$\frac{|g(x-y+te_1) - g(x-y)|}{|t|} \leq \max_{\bar{B}_{3R}} |\partial_{x_1} g| =: M.$$

So the integrand above satisfies

$$|\text{integrand}| \leq M|f(y)|.$$

Differentiation rule for convolution

Proof

- So we have, for $|t| \leq R$,

$$D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y) \frac{g(x-y+te_1) - g(x-y)}{t}\, dy$$

  where

  - ⋆ integrand $\to f(y)\partial_{x_1} g(x-y)$ as $t \to 0$.
  - ⋆ $|\text{integrand}| \leq M|f(y)|$, which belongs to $L^1(\{|x-y| \leq 2R\})$, as $f \in L^p(\mathbb{R}^n)$.

- By Lebesgue's dominated convergence theorem, we thus have

$$\lim_{t\to 0} D.Q.(x,t) = \int_{|x-y|\leq 2R} f(y)\partial_{x_1} g(x-y)\, dy$$
$$= \int_{\mathbb{R}^n} f(y)\partial_{x_1} g(x-y)\, dy = (f * \partial_{x_1} g)(x).$$

Differentiation rule for convolution

Proof

- We conclude that $\partial_{x_1}(f * g)$ exists and is equal to $f * \partial_{x_1} g$.
- By the previous lemma, we have that $f * \partial_{x_1} g$ is continuous. So $\partial_{x_1}(f * g)$ is continuous. Applying this to all partial derivatives, we conclude that $f * g \in C^1(\mathbb{R}^n)$.