of x, and the square of this sum instead of xx, and its cube instead of x^3, and so on for others, if it is x that I want to eliminate; or [343] instead, if it is y, by putting in its place $v + \sqrt{ss - xx}$, and the square, cube, etc. of this sum in place of yy, or y^3 etc. In such a way that there always remains after this an equation in which there is no more than a single unknown quantity x or y.

If CE is an ellipse, and MA is the segment of its diameter to which CM is applied as an ordinate, and which has r for its *latus rectum*, and q for the transverse diameter, one has by Proposition 13 of Book I of Apollonius $xx = ry - \frac{r}{q}yy$, where eliminating xx there remains $ss - vv + 2vy - yy = ry - \frac{r}{q}yy$. Or rather, $yy\frac{+qry-2qvy+qvv-qss}{q-r}$ is equal to nothing. For it is better in this case to consider the whole sum together in this way, than to make one part equal to another.

[...]

Descartes went on to explain that the equation thus found is to be used to discover v or s. Further, he argued, if P is the point required, then the circle through C with P as the centre will touch the curve without cutting it, and so the equation between y, v, and s will have two equal roots.

[347] As if, for example, I say that the first equation found above, namely $yy\frac{+qry-2qvy+qvv-qss}{q-r}$ must take the same form as that produced by making e equal to y and multiplying $y - e$ by itself, from which there comes $yy - 2ey + ee$, so that one may compare each of their terms separately, and say that because the first, which is yy is just the same in one as in the other, the second which is in one $\frac{qry-2qvy}{q-r}$, is equal to the second of the other, which is $-2ey$, from which, seeking the quantity v, which is the line PA, we have $v = e - \frac{r}{q}e + \frac{1}{2}r$, or rather, because we have supposed e equal to y, we have $v = y - \frac{r}{q}y + \frac{1}{2}r$. And [348] thus one may find s from the third term $ee = \frac{qvv-qss}{q-r}$ but because the quantity v sufficiently determines the point P, which is the only one we were seeking, one has no need to go further.

3.2 METHODS OF QUADRATURE

3.2.1 Fermat's quadrature of higher hyperbolas, early 1640s

A problem of much wider concern than finding tangents was that of quadrature: literally, finding a square equal to a given space or, in modern terms, finding an area. Attempts at quadrature were many and varied, giving rise to numerous special methods for special cases. Here we can present only a few examples, chosen to illustrate some of the more important ideas that were beginning to emerge during the seventeenth century. For ease of reading in what follows we will borrow modern notation for summations and for equations of curves.

By 1636, both Roberval and Fermat knew that the value of $\sum_{x=0}^{X} x^n$ is approximately $\frac{X^n}{n+1}$ when n is a positive integer and x is taken at sufficiently small intervals between 0 and X. Both used this relationship to find the quadrature of curves of the form $y = x^n$. Fermat continued to explore such questions privately during the early 1640s, and appears to have found methods of quadrature also for curves of the form $y^m = x^n$ (higher parabolas) and $y^m = x^{-n}$ (higher hyperbolas), but unfortunately it is impossible to date his work precisely. We know that he corresponded with Torricelli on the subject in 1644, but the letters themselves are now lost.

Only in 1658–59 did Fermat bring his results together in a treatise headed 'De aequationum localium transmutatione …cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus' ('On the transformation of equations of place …to which is adjoined the use of geometric progressions for squaring infinite parabolas and hyperbolas'). This was almost certainly written in response to Wallis's *Arithmetica infinitorum* of 1656, which treated similar problems in a rather different way. By the late 1650s, however, Fermat's results were no longer new, and the treatise remained unpublished until 1679, long after his death.

The procedure below is from the opening of 'De aequationum localium'. It can be applied to any curve of the form $y = x^{-n}$ except when $n = 1$, the case Fermat described as the hyperbola of Apollonius. The method is based on dividing the required area into strips whose bases increase in geometric progression. Because of the rapid fall of the curve the areas of the corresponding rectangles decrease, also in geometric progression. Fermat knew (from Euclid IX.35) how to sum a finite geometric progression and, like Viète before him, extended the result to an infinite progression by taking the 'last' term to be zero.

In many respects Fermat's proof remains strongly reminiscent of the Greek mathematics to which he made such frequent reference: it is entirely geometric, and couched throughout in the Euclidean language of ratio. In other ways, however, he went far beyond the classical methods of exhaustion and contradiction. In August 1657 he had complained that Wallis could just as well have handled quadratures in the traditional Archimedean way,[2] yet in his own treatment he discussed the Archimedean method only to dismiss it and move on. His summation of a geometric progression with an infinite number of terms was an idea learned from Viète, not from Euclid. And just as in his tangent method he had introduced quantities that were allowed to vanish once they had served their purpose, here too he used a similar procedure: the parallelogram *EGH* plays a crucial role in his argument, but when no longer needed it simply 'goes to nothing' ('abit in nihilum').

2. Wallis 1658, letter 12.

Fermat's quadrature of a hyperbola

from Fermat, *Varia opera*, 1679, 44–46

DE ÆQUATIONUM LOCALIUM TRANS

mutatione, & emendatione, ad multimodam curvilineorum inter ſe, vel cum rectilineis comparationem.

CVI ANNECTITVR

PROPORTIONIS GEOMETRICÆ in quadrandis infinitis parabolis & hyperbolis uſus.

IN unica paraboles quadraturâ proportionem geometricam uſurpavit Archimedes. In reliquis quantitatum heterogenearum comparationibus, arithmeticæ dumtaxat proportioni ſeſe adſtrinxit. An ideo quia proportionem geometricam minùs τετραγωνίζουσαν eſt expertus? An verò quia peculiare ab illa proportione petitum artificium ad quadrandam primariam parabolam, ad ulteriores derivari vix poteſt? Nos certè hujuſmodi proportionem quadrationum feraciſſimam & agnoſcimus, & experti ſumus, & inventionem noſtram quæ eadem omnino methodo & parabolas & hyperbolas quadrat, recentioribus geometris haud illibenter impertimur.

Unico quod notiſſimum eſt proportionis geometricæ attributo, tota hæc methodus innititur.

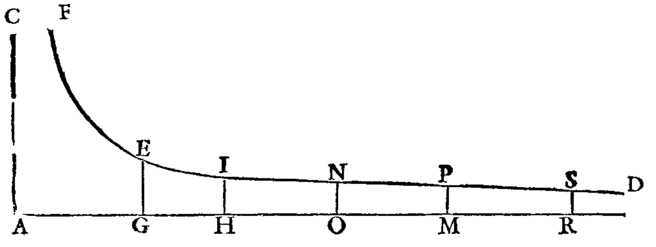
Theorema hoc eſt: Datâ quavis proportione geometricâ cujus termini decreſcant in infinitum, eſt ut differentia terminorum progreſſionem conſtituentium, ad minorem terminum, ita maximus progreſſionis terminus ad reliquos omnes in infinitum ſumptos.

Hoc poſito, proponantur primo hyperbolæ quadrandæ. Hyperbolas autem definimus infinitas diverſæ ſpeciei curvas, ut D S E F, quarum hæc eſt proprietas, ut poſitis in quolibet angulo dato R A C, ipſarum aſymptotis rectis A R, A C, in infinitum, ſi placet, non ſecùs ac ipſa curva extendendis, & ductis uni aſymptotωn parallelis rectis quibuſlibet G E, H I, O N, M P, R S, &c. ſit ut poteſtas quædam rectæ A H, ad poteſtatem ſimilem rectæ A G, ita poteſtas rectæ G E, vel ſimilis vel diverſa à præcedente, ad poteſtatem ipſi homogeneam rectæ H I, poteſtates autem intelligimus, non ſo-

Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum



O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad-ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

G E, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A. G A, ſunt proportionales ex conſtructione.

F 3

46 **Varia Opera**

Ergo parallelogramma in infinitum ſumpta ſub G E, in G H, ſub H I, in H O, ſub O N, in O M, &c. erunt ſemper continuè proportionalia in ratione rectæ H A, ad G A. Eſt igitur ex theoremate hujus methodi conſtitutivo ut G H, differentia terminorum rationis ad minorem terminum G A, ita primus parallelogrammorum progreſſionis terminus, hoc eſt parallelogrammum ſub E G, in G H, ad reliquos in infinitum parallelogrammos, hoc eſt ex adæquatione Archimedæa ad figuram ſub H I, aſymptoto H R, & curvâ in I N D, in infinitum extendendâ contentam. Sed ut H G, ad G A, ita ſumptâ communi latitudine recta G E, parallelogrammum ſub G E, in G H, ad parallelogrammum ſub G E in G A. Eſt igitur ut parallelogrammum ſub G E, in G H, ad figuram illam infinitam, cujus baſis H I, ita idem parallelogrammum ſub G E, in G H, ad parallelogrammum ſub G E, in G A, ergo parallelogrammum ſub G E, in G A, quod eſt ſpatium rectilineum datum, adæquatur figuræ prædictæ. Cui ſi addas parallelogrammum ſub G E, in G H, quod propter minutiſſimos πμαχισμὸς evaneſcit & abit in nihilum, ſupereſt veriſſimum, & Archimedæâ licet prolixiore demonſtratione facillimè firmandum, parallelogrammum A E, in hac hyperbolæ ſpecie, æquari figuræ ſub baſe G E, aſymptoto G R, & curvâ E D, in infinitum producendâ contentæ. Nec operoſum ad omnes omnino hujuſmodi hyperbolas, unâ, ut diximus, demptâ, inventionem extendere.

TRANSLATION

ON THE TRANSFORMATION AND EMENDATION OF EQUATIONS OF PLACE

in order to compare curves in various ways with each other, or with straight lines

TO WHICH IS ADJOINED

THE USE OF GEOMETRIC PROGRESSIONS

in the quadrature of infinite parabolas or hyperbolas

Archimedes made use of geometric progressions only for the quadrature of one parabola. In the remaining comparisons of heterogeneous quantities he restricted himself merely to arithmetic progressions. Whether because he found geometric progressions less appropriate? Or because the required method with the particular progression used for squaring the first parabola could scarcely be extended to the others? I have certainly recognized, and proved, progressions of this kind very productive for quadratures, and my discovery, by which one may square both parabolas and hyperbolas by exactly the same method, I by no means unwillingly communicate to more modern geometers.

I attribute to geometric progressions only what is very well known, on which this whole method is based.

The theorem is this: Given any geometric progression whose terms decrease infinitely, as the difference of two [consecutive] terms constituting the progression is to the smaller of them, so is the greatest term of the progression to all the rest taken infinitely.

This established, there is proposed first the quadrature of hyperbolas. Moreover we define hyperbolas as infinite curves of various kinds, like *DSEF*, of which this is a property, that having placed at any given angle *RAC* its asymptotes, *AR*, *AC*, extended infinitely if one pleases but not cut by the curve, and taking whatever straight lines, *GE*, *HI*, *ON*, *MP*, *RS*, etc. parallel to one asymptote, we suppose that a certain power of the line *AH* to the same power of the line *AG* is as a power of the line *GE*, whether the same or different from the preceding one, to that same power of the line *HI*; moreover we understand the powers to be not [45] only squares, cubes, square-squares, etc. of which the exponents are 2, 3, 4, etc. but also simple lines, whose power is one. I say, therefore, that all hyperbolas of this kind indefinitely, with one exception, which is that of Apollonius, or the first, can be squared with the help of the same and always applicable method of geometric progressions.

Let there be, if one likes, a hyperbola of which it is the property that the square of the line *HA* to the square of the line *AG* is always as the line *GE* to the line *HI*, and that the square of *OA* to the square of *AH* is as the line *HI* to the line *ON*, etc. I say that the infinite space whose base is *GE*, and with the curve *ES* for one side, but for the other the infinite asymptote *GOR*, is equal to a given rectilinear space. It is supposed that the terms of a geometric progression can be extended infinitely, of which the first is *AG*, the second *AH*, the third *AO*, etc. infinitely, and these approach each other by approximation as closely as is needed, so that by the method of Archimedes the parallelogram made by *GE* and *GH* adequates, as Diophantus says, to the irregular four-sided shape *GHE*, or very nearly equals.

GE times *GH*.

Likewise, the first of the straight line intervals of the progression *GH*, *HO*, *OM*, and so on, are similarly very nearly equal amongst themselves, so that we can conveniently use the method of exhaustion, and by Archimedean circumscriptions and inscriptions the ratio to be demonstrated can be established, which it is sufficient to have shown once, nor do I wish to repeat or insist more often on a method already sufficiently known to any geometer.

This said, since *AH* to *AO* is as *AG* to *AH*, so also will *AO* to *AM* be as *AG* to *AH*. So also will be the interval *GH* to *HO*, and the interval *HO* to *HM*, etc. Moreover the parallelogram made by *EG* and *GH* will be to the parallelogram made by *HI* and *HO*, as the parallelogram made by *HI* and *HO* to the parallelogram made by *NO* and *OM*, for the ratio of the parallelogram made by *GE* and *GH* to the parallelogram made by *HI* and *HO* is composed from the ratio of the line *GE* to the line *HI*, and from the ratio of the line *GH* to the line *HO*; and as *GH* is to *HO*, so is *AG* to *AH*, as we have shown. Therefore the ratio of the parallelogram made by *EG* and *GH* to the parallelogram made by *HI* and *HO* is composed from the ratio *GE* to *HI*, and from the ratio *AG* to *AH*, but as *GE* is to *HI* so by construction will be the square of *HA* to the square of *GA*, or because of proportionality, as the line *AO* to the line *GA*. Therefore the ratio of the parallelogram made by *EG* and *GH* to the parallelogram made by *HI* and *HO*, will be composed of the ratios *AO* to *AG*, and *AG* to *AH*; but the ratio *AO* to *AH* is composed of these two. Therefore the parallelogram made by *GE* and *GH* is to the parallelogram made by *HI* and *HO*, as *OA* to *HA*; or as *HA* to *AG*.

Similarly it can be proved that the parallelogram made by *HI* and *HO* is to the parallelogram made by *ON* and *OM*, as *AO* to *HA*, but the three lines that constitute the ratios of the parallelogams, namely *AO*, *HA*, *GA*, are proportionals by construction.