[46] Therefore the parallelograms made by *GE* and *GH*, by *HI* and *HO*, by *ON* and *OM*, etc. taken indefinitely, will always be continued proportionals in the ratio of the lines *HA* to *GA*. Therefore, from the theorem that is the foundation of this method, as *GH*, the difference of the terms of the progression, is to the smaller term *GA*, so will be the first term of the progression of parallelograms, that is, the parallelogram made by *EG* and *GH*, to the rest of the parallelograms taken infinitely, that is, by the adequation of Archimedes, to the space contained by *HI*, the asymptote *HR*, and the curve *IND* extended infinitely. But as *HG* is to *GA* so, taking as a common side the line *GE*, is the parallelogram made by *GE* and *GH* to the parallelogram made by *GE* and *GA*. Therefore, as the parallelogram made by *GE* and *GH* is to that infinite figure whose base is *HI*, so is the same parallelogram made by *GE* and *GH* to the parallelogram made by *GE* and *GA*; therefore the parallelogram made by *GE* and *GA*, which is the given rectilinear space, adequates to the aforesaid figure. To which if there is added the parallelogram made by *GE* and *GH*, which on account of the minute divisions vanishes and goes to nothing, there remains the truth, which may be easily confirmed by a more lengthy Archimedean demonstration, that the parallelogram *AE* in this kind of hyperbola, is equal to the space contained between the base *GE*, the asymptote *GR*, and the curve *ED*, infinitely produced. Nor is it onerous to extend this discovery to all hyperbolas of this kind, except, as I said, one.

3.2.2 Brouncker and the rectangular hyperbola, *c.* 1655

The quadrature of the Apollonian, or rectangular, hyperbola ($y = x^{-1}$) had eluded Fermat, but some partial results had been found by de Saint Vincent as early as 1625. When de Saint Vincent's work was eventually published in his massive *Opus geometricum* in 1647, his fellow Jesuit Alphonse Antonio de Sarasa noted that certain areas under the hyperbola are related to each other in the same way as logarithms, but at this stage this was no more than an empirical observation.

A numerical quadrature of the hyperbola was finally discovered by William Brouncker in the early 1650s while he was working with Wallis on the related problem of the quadrature of the circle (see 3.2.3). Brouncker's quadrature was published in 1668 in the third volume of the *Philosophical transactions of the Royal Society*, the first mathematical result to be published in a scientific journal. (Brouncker himself was the first president of the Royal Society, which had been founded in 1660.)

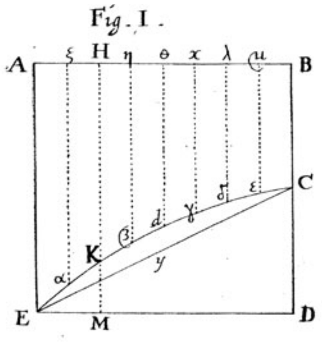
Though an able mathematician, Brouncker was never forthcoming about his methods, and offered only diagrams and results, without any intermediate calculations. He did however, offer the first and almost only seventeenth-century attempt at a convergence proof.

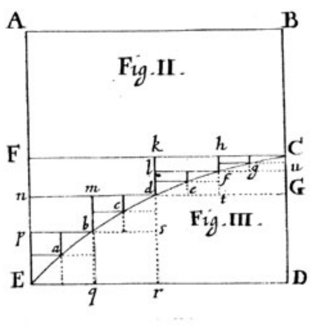
Brouncker's quadrature of the hyperbola

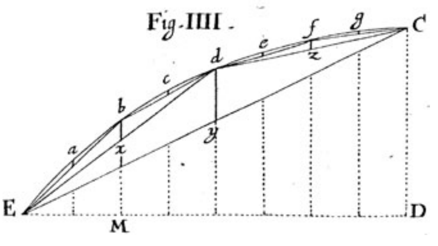
from Brouncker, 'The squaring of the hyperbola by an infinite series of rational numbers', *Philosophical transactions of the Royal Society*, 3 (1668), 645–647

Notation

Although Brouncker adopted Descartes' superscript notation for powers, he retained Harriot's *a* rather than Descartes' *x*.







(645) *Numb.* 34.

PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola, *by an infinite ſeries of Rational Numbers, together with its Demonſtration, by that Eminent Mathematician, the Right Honourable the Lord Viſcount* Brouncker.

WHat the Acute Dr. *John Wallis* had intimated, ſome years ſince, in the Dedication of his Anſwer to M. *Meibomius de proportionibus, vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may ſee performed in the ſubjoyned operation, which its Excellent Author was now pleaſed to communicate, as followeth in his own words;

Zzz My

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My Method for Squaring the Hyperbola *is this*:

LEt AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Lett:r x every where stands for Multiplication.

Supposing the Reader knows, that EA. αζ. KH. βη. dθ. γκ. δλ. εμ. CB.&c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmetice proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisij*:)

I say $$\left.\begin{array}{l} ABCdEA = \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10} \&c. \\ EdCDE = \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11} \&c. \\ EdCyE = \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10} \&c. \end{array}\right\} \textit{in infinitum.}$$

For (*in Fig.* 2, & 3) *the Parallelog.*

$$CA = \frac{1}{1\times2}$$

$dD = \frac{1}{2\times3}$	$dF = \frac{1}{3\times4}$
$br = \frac{1}{4\times5}$	$bn = \frac{1}{5\times6}$
$fG = \frac{1}{6\times7}$	$fk = \frac{1}{7\times8}$
$aq = \frac{1}{8\times9}$	$ap = \frac{1}{9\times10}$
$cs = \frac{1}{10\times11}$	$cm = \frac{1}{11\times12}$
$et = \frac{1}{12\times13}$	$el = \frac{1}{13\times14}$
$gu = \frac{1}{14\times15}$	$gh = \frac{1}{15\times16}$
&c.	&c.

And (*in Fig.* 4.) *the Triangl.*

$$EdC = \frac{1}{2\times3\times4} = \frac{\square dD - \square dF}{2}$$
$$Ebd = \frac{1}{4\times5\times6} = \frac{\square br - \square bn}{2}$$
$$dfC = \frac{1}{6\times7\times8} = \frac{\square fG - \square fk}{2}$$
$$Eab = \frac{1}{8\times9\times10} = \frac{\square aq - \square ap}{2}$$
$$bcd = \frac{1}{10\times11\times12} = \frac{\square cs - \square cm}{2}$$
$$def = \frac{1}{12\times13\times14} = \frac{\square et - \square el}{2}$$
$$fgC = \frac{1}{14\times15\times16} = \frac{\square gu - \square gh}{2}$$

&c.

Note.

$CA = dD + dF$

$dD = br + bn$

$dF = fG + fk$

$br = aq + ap$

$bn = cs + cm$

$fG = et + el$

$fk = gu + gh$

&c.

And

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And that therefore in the firſt ſeries half the firſt term is greater than the ſum of the two next, and half this ſum of the ſecond and third greater than the ſum of the four next, and half the ſum of thoſe four greater than the ſum of the next eight, *&c. in infinitum.* For $\frac{1}{2}$ *dD* = *br* + *bn*; but *bn* > *fG*, therefore $\frac{1}{2}$ *dD* > *br* + *fG*, *&c.* And in the *ſecond* ſeries half the firſt term is leſs then the ſum of the two next, and half this ſum leſs then the ſum of the four next, *&c. in infinitum.* second

That the firſt *ſeries* are the *even* terms, *viz.* the 2[d], 4[th], 6[th], 8[th], 10[th], *&c.* and the *ſecond*, the *odd*, *viz.* the 1[st], 3[d], 5[th], 7[th], 9[th], *&c.* of the following ſeries, *viz.* $\frac{1}{1\times2}$ $\frac{1}{2\times3}$ $\frac{1}{3\times4}$ $\frac{1}{4\times5}$ $\frac{1}{5\times6}$ $\frac{1}{6\times7}$ *&c. in infinitum* = 1. Whereof *a* being put for the number of terms taken at pleaſure, $\frac{1}{a+a}$ is the laſt, $\frac{a}{a+1}$ is the ſum of all thoſe terms from the beginning, and $\frac{1}{a+1}$ the ſum of the reſt to the end.

That $\frac{1}{4}$ of the firſt terme in the *third* ſeries is leſs than the ſum of the two next, and a quarter of this ſum, leſs than the ſum of the four next, and one fourth of this laſt ſum leſs than the next eight, I thus demonſtrate.

Let a = the 3[d] or laſt number of any term of the firſt Column, *viz.* of Diviſors,

$$\frac{1}{a \times a-1 \times a-2} = \frac{1}{a^3-3a^2+2a} = \frac{16a^3-48a^2+56a-24}{16a^6-96a^5+232a^4-288a^3+184a^2-48a} = A$$

$$\left.\begin{array}{l} \frac{1}{2a \times 2a-1 \times 2a-2} = \frac{1}{8a^3-12a^2+4a} \\ \frac{1}{2a-2 \times 2a-3 \times 2a-4} = \frac{1}{8a^3-36a^2+52a-24} \end{array}\right\} = \frac{16a^3-48a^2+56a-24}{64a^6-384a^5+880a^4-960a^3+496a-96} = B$$

$$\frac{64a^6-384a^5+928a^4-1152a^3+736a^2-192a}{64a^6-384a^5+880a^4-960a^3+496a^2-96a} \times \tfrac{1}{4} A \gtrless B. \qquad =$$

And $48a^4-192a^3+240a^2-96a$ = *Exceſs of the Numerator above Denomin.*

$$\begin{array}{lll} \text{But — The affirm.} & > & \text{the Negat.} \\ \text{That is, } 48a^4+240a^2 & > & 192a^3+96a \\ \text{Becauſe } a^4+5a^2 & > & 4a^3+2a \\ a^3+5a & > & 4a^2+2 \end{array} \Bigg\} \text{ if } a > 2.$$

Therefore B > $\frac{1}{4}$ A.

Therefore $\frac{1}{4}$ of any number of A. or Terms, is leſs than their ſo many reſpective B. that is, than twice ſo many of the next Terms. *Quod, &c.*

Aaaa 2 By