# C4.3 Functional Analytic Methods for PDEs
# Lecture 4

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# In the last lecture

- Weak and weak* convergence in Lebesgue spaces.
- Continuity property of translation operators in $L^p$.
- Convolution. Young's inequality.
- Differentiation rule for convolution.

# This lecture

- Approximation of identity in Lebesgue spaces.
- Density by smooth functions.
- Pre-compactness criteria.

# Approximation of identity

- A family of "kernels" $\{\varrho_\varepsilon : \mathbb{R}^n \to \mathbb{R}\}_{\varepsilon > 0}$ is called an approximation of identity if

$$f * \varrho_\varepsilon \text{``} \to \text{''} f \text{ as } \varepsilon \to 0,$$

  where the meaning of the convergence depends on the context.
- Loosely speaking, it means that the operators $T_\varepsilon$ defined by $T_\varepsilon f = f * \varrho_\varepsilon$ "approximates" the identity operator.

# Approximation of identity in continuous settings

## Theorem (Approximation of identity)

*Let $\varrho$ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

*If $f \in C(\mathbb{R}^n)$, then $f * \varrho_\varepsilon$ converges uniformly on compact subsets of $\mathbb{R}^n$ to $f$.*

More on terminologies:

- A family $(\varrho_\varepsilon)$ as in the statement is called a family of 'mollifiers'.
- The family $(f * \varrho_\varepsilon)$ is called a regularization of $f$ by mollification. Note that since $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have that $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n)$.

# Approximation of identity in continuous settings

Proof:

- Let us first consider pointwise convergence, i.e. for every $x$ there holds:

$$(f * \varrho_\varepsilon)(x) = \int_{\mathbb{R}^n} f(y)\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon\to 0} f(x).$$

- The idea is to convert $f(x)$ into an integral as well. For this we use the identity

$$\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} \varrho_\varepsilon(z)\,dz = \int_{\mathbb{R}^n} \varrho(w)\,dw = 1.$$

  Hence

$$f(x) = \int_{\mathbb{R}^n} f(x)\varrho_\varepsilon(x-y)\,dy.$$

# Approximation of identity in continuous settings

Proof:

- So we need to show

$$\int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \xrightarrow{\varepsilon \to 0} 0.$$

- By hypotheses, $\varrho$ vanishes outside of some ball $B_R$ centered at the origin. So $\varrho_\varepsilon(x-y) = 0$ when $|x-y| \geq \varepsilon R$. It follows that

$$\Big| \int_{\mathbb{R}^n} [f(x) - f(y)]\varrho_\varepsilon(x-y)\,dy \Big|$$
$$\leq \sup_{\{y : |x-y| \leq \varepsilon R\}} |f(x) - f(y)| \int_{|x-y| \leq \varepsilon R} \varrho_\varepsilon(x-y)\,dy$$
$$= \sup_{\{y : |x-y| \leq \varepsilon R\}} |f(x) - f(y)| \xrightarrow{\varepsilon \to 0} 0.$$

# Approximation of identity in continuous settings

Proof:

- Now we turn to prove the uniform convergence on compact sets, i.e. for every given compact set $K$, we need to show

$$\sup_{x \in K} \Big|(f * \varrho_\varepsilon)(x) - f(x)\Big| \xrightarrow{\varepsilon \to 0} 0.$$

  As before, this is equivalent to

$$\sup_{x \in K} \Big| \int_{\mathbb{R}^n} [f(x) - f(y)] \varrho_\varepsilon(x - y)\, dy \Big| \xrightarrow{\varepsilon \to 0} 0,$$

  which can be turned into

$$\sup_{x \in K} \Big| \int_{\{y : |x-y| \le \varepsilon R\}} [f(x) - f(y)] \varrho_\varepsilon(x - y)\, dy \Big| \xrightarrow{\varepsilon \to 0} 0,$$

# Approximation of identity in continuous settings

Proof:

- We need to show

$$A_\varepsilon := \sup_{x \in K} \left| \int_{\{y : |x-y| \leq \varepsilon R\}} [f(x) - f(y)] \varrho_\varepsilon(x-y)\, dy \right| \stackrel{\varepsilon \to 0}{\longrightarrow} 0,$$

- In the same way as before, we have

$$A_\varepsilon \leq \sup_{x \in K} \sup_{\{y : |x-y| \leq \varepsilon R\}} |f(x) - f(y)|.$$

- Note that if $K \subset B_{R'}$, $\varepsilon \leq 1$, $x \in K$ and $|x-y| \leq \varepsilon R$, then
  - ⋆ $|x| \leq R' \leq R + R'$,
  - ⋆ $|y| \leq |x| + |y-x| \leq R + R'$.

  So

$$A_\varepsilon \leq \sup_{\{|x|,|y| \leq R+R', |x-y| \leq \varepsilon R\}} |f(x) - f(y)| \stackrel{\varepsilon \to 0}{\longrightarrow} 0,$$

  in view of the uniform continuity of $f$ on $\overline{B_{R+R'}}$.

# Approximation of identity in Lipschitz settings

## Theorem (Approximation of identity)

*Let $\varrho$ be a non-negative function in $C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

*If $f \in C^{0,1}(\mathbb{R}^n)$, i.e. there exists $L \geq 0$ such that*

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n,$$

*then, for some constant $C > 0$ depending only on the choice of $\varrho$,*

$$\sup_{x \in \mathbb{R}^n} |f * \varrho_\varepsilon(x) - f(x)| \leq CL\varepsilon.$$

# Approximation of identity in Lipschitz settings

Proof: Following the same argument as before, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n}\Big|(f*\varrho_\varepsilon)(x) - f(x)\Big| &= \sup_{x\in\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}[f(x)-f(y)]\varrho_\varepsilon(x-y)\,dy\Big| \\
&\leq \sup_{x\in\mathbb{R}^n}\sup_{\{y:|x-y|\leq\varepsilon R\}}|f(x)-f(y)| \\
&\leq \sup_{x\in\mathbb{R}^n}\sup_{\{y:|x-y|\leq\varepsilon R\}} L|x-y| \\
&\leq L\varepsilon R.
\end{aligned}$$

# Approximation of identity in $L^p$ settings

## Theorem (Approximation of identity)

*Let $\varrho$ be a non-negative function in $L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n} \varrho\Big(\frac{x}{\varepsilon}\Big).$$

*If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then*

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

# $f * \varrho_\varepsilon \not\to f$ in $L^\infty$

## Remark

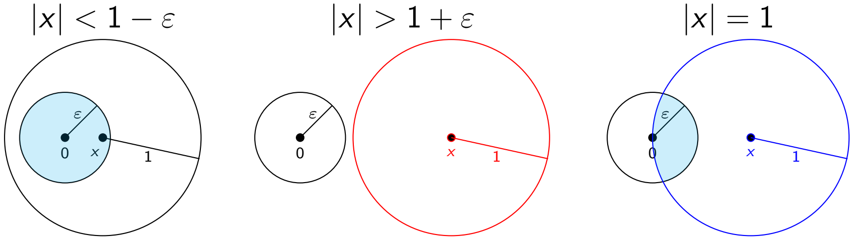
*There exist $f \in L^\infty(\mathbb{R}^n)$ and $\varrho \in C_c^\infty(B_1(0))$ such that $f * \varrho_\varepsilon$ does not converge to $f$ in $L^\infty$.*

- Take $f = \chi_{B_1(0)}$.
- Then

$$\begin{aligned} f * \varrho_\varepsilon(x) &= \int_{B_1(0)} \varrho_\varepsilon(x - y)\, dy \\ &= \int_{B_1(x)} \varrho_\varepsilon(z)\, dz \\ &= \int_{B_1(x) \cap B_\varepsilon(0)} \varrho_\varepsilon(z)\, dz. \end{aligned}$$

# $f * \varrho_\varepsilon \not\to f$ in $L^\infty$

- $f * \varrho_\varepsilon(x) = \int_{B_1(x)\cap B_\varepsilon(0)} \varrho_\varepsilon(z)\,dz$.



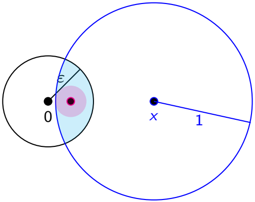

$f * \varrho_\varepsilon(x) = 1$   $f * \varrho_\varepsilon(x) = 0$   $f * \varrho_\varepsilon(x) \in [0,1]$ $\to \frac{1}{2}$ in symmetry, i.e. $\varrho = \varrho(|x|)$

# $f * \varrho_\varepsilon \not\to f$ in $L^\infty$

- We now take some $\varrho$ of the form $\varrho(x) = \varrho(|x|)$ such that, in addition to the condition $\|\varrho\|_{L^1} = 1$, we have

$$\int_{B_{1/4}(p)} \varrho(z)\,dz = c_0 \in (0,1) \text{ for all } |p| = 1/2.$$

- Consider $1 < |x| < 1 + \varepsilon/4$.




  - ⋆ $B_1(x) \cap B_\varepsilon(0)$ contains a ball $B_{\varepsilon/4}(p_\varepsilon)$ with $|p_\varepsilon| = \varepsilon/2$.
  - ⋆ So $f * \varrho_\varepsilon(x) \geq \int_{B_{\varepsilon/4}(p_\varepsilon)} \varrho_\varepsilon(z)\,dz = c_0 \in (0,1)$.
  - ⋆ As $f(x) = 0$ here, we thus have

$$\|f * \varrho_\varepsilon - f\|_{L^\infty} \geq c_0 \not\to 0.$$

# Approximation of identity in $L^p$ settings

## Theorem (Approximation of identity)

*Let* $\varrho$ *be a non-negative function in* $L^1(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varrho = 1$. *For* $\varepsilon > 0$, *let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

*If* $f \in L^p(\mathbb{R}^n)$ *for some* $1 \leq p < \infty$, *then*

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

# Approximation of identity in $L^p$ settings

Proof

- Let $f_\varepsilon(x) := f * \varrho_\varepsilon(x)$. Then

$$f_\varepsilon(x) := f * \varrho_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\varrho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)\varrho_\varepsilon(y)\,dy.$$

- Recall that, as $\int_{\mathbb{R}^n} \varrho_\varepsilon = 1$, we have

$$f(x) = \int_{\mathbb{R}^n} f(x)\varrho_\varepsilon(y)\,dy.$$

- Hence

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|dy \\ &= \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|^{\frac{1}{p}}|\varrho_\varepsilon(y)|^{\frac{1}{p'}}\,dy. \end{aligned}$$

# Approximation of identity in $L^p$ settings

Proof

- $|f_\varepsilon(x) - f(x)| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varrho_\varepsilon(y)|^{\frac{1}{p}}|\varrho_\varepsilon(y)|^{\frac{1}{p'}}\,dy.$
- Applying Hölder's inequality, the above is less than or equal to

$$\leq \Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p}}\Big\{\int_{\mathbb{R}^n} |\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p'}}$$
$$= \Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\Big\}^{\frac{1}{p}}.$$

- Integrating and using Tonelli's theorem,

$$\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p|\varrho_\varepsilon(y)|\,dy\,dx$$
$$= \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)|\Big\{\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p\,dx\Big\}dy.$$

# Approximation of identity in $L^p$ settings

Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \Big\{ \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \, dx \Big\} dy.$
- In other words,

$$\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy.$$

- If we had that $Supp(\varrho) \subset B_R$, then $Supp(\varrho_\varepsilon) \subset B_{\varepsilon R}$, and so

$$\begin{aligned}\|f_\varepsilon - f\|_{L^p}^p &\leq \sup_{|y| \leq \varepsilon R} \|\tau_{-y} f - f\|_{L^p}^p \int_{B_{\varepsilon R}} |\varrho_\varepsilon(y)| dy \\ &= \sup_{|y| \leq \varepsilon R} \|\tau_{-y} f - f\|_{L^p}^p \xrightarrow{\varepsilon \to 0} 0,\end{aligned}$$

  in view of the theorem on the continuity of the translation operator in $L^p$.

# Approximation of identity in $L^p$ settings

Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- In the general case where $\varrho$ may or may not have compact support, we argue as follows: For every fixed $\hat{R} > 0$,

$$\begin{aligned}
&\int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \\
&\quad \leq \sup_{|y| \leq \varepsilon \hat{R}} \|\tau_{-y} f - f\|_{L^p}^p \int_{B_{\varepsilon \hat{R}}} |\varrho_\varepsilon(y)| dy \\
&\quad \leq \sup_{|y| \leq \varepsilon \hat{R}} \|\tau_{-y} f - f\|_{L^p}^p \xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

# Approximation of identity in $L^p$ settings

Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- $\forall\ \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- On the other hand,

$$\begin{aligned}\int_{|y| \geq \varepsilon \hat{R}} & |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \\ & \leq \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| (\|\tau_{-y} f\|_{L^p} + \|f\|_{L^p})^p dy \\ & = 2^p \|f\|_{L^p}^p \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| dy = 2^p \|f\|_{L^p}^p \int_{|z| \geq \hat{R}} |\varrho(z)| dz.\end{aligned}$$

- As $\varrho \in L^1(\mathbb{R}^n)$, we have by Lebesgue's dominated convergence theorem that

$$\lim_{\hat{R} \to \infty} \int_{|z| \geq \hat{R}} |\varrho(z)| dz = 0.$$

# Approximation of identity in $L^p$ settings

Proof

- $\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy$.
- $\forall\ \hat{R}$, $\lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- $\forall\ \varepsilon$, $\lim_{\hat{R} \to \infty} \int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy = 0$.
- We are ready to wrap up the proof: Fix some $\eta > 0$ and select some large $\hat{R}$ so that

$$\int_{|y| \geq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta/2.$$

- Then we select small $\varepsilon_0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\int_{|y| \leq \varepsilon \hat{R}} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta/2.$$

# Approximation of identity in $L^p$ settings

Proof

- For $\varepsilon < \varepsilon_0$,

$$\|f_\varepsilon - f\|_{L^p}^p \leq \int_{\mathbb{R}^n} |\varrho_\varepsilon(y)| \|\tau_{-y} f - f\|_{L^p}^p dy \leq \eta.$$

- As $\eta$ is arbitrary, we conclude that $\|f_\varepsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$.

# $\overline{C_c^\infty(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$

## Theorem

*For $1 \le p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof

- Fix $f \in L^p(\mathbb{R}^n)$. We need to produce $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^p$.
- If $f$ has compact support, say $Supp(f) \subset B_R$, this follows from the previous theorem:
  - ⋆ Take a smooth non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and define the mollifiers $\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho(x/\varepsilon)$.
  - ⋆ Then $f_k := f * \varrho_{1/k} \in C^\infty$ and $f_k \to f$ in $L^p$.
  - ⋆ Recall that $f_k(x) = \int_{\mathbb{R}^n} f(y)\varrho_{1/k}(x-y)\,dy$ , and observe that for $|x| > R + 1/k$, then, by triangle inequality, $|y| > R$ or $|x-y| > 1/k$. So $f(y)\varrho_{1/k}(x-y) \equiv 0$ for those $x$. So $Supp(f_k) \subset B_{R+1/k}$, and $f_k \in C_c^\infty(\mathbb{R}^n)$.

# $\overline{C_c^\infty(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$

Proof

- In general, we produce $f_k \in C_c^\infty(\mathbb{R}^n)$ with $\|f_k - f\|_{L^p} \leq 1/k$ as follows:
  - ⋆ Note that, as $R \to \infty$, $f\chi_{B_R} \to f$ in $L^p$ by Lebesgue's dominated convergence theorem.
  - ⋆ So we can pick large $R_k$ such that $\|f\chi_{B_{R_k}} - f\|_{L^p} \leq \frac{1}{2k}$.
  - ⋆ Now $f\chi_{B_{R_k}}$ has compact support, so by the previous consideration, there exists a function $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $\|f\chi_{B_{R_k}} - f_k\|_{L^p} \leq \frac{1}{2k}$.
  - ⋆ Then $\|f_k - f\|_{L^p} \leq 1/k$ and so $f_k \to f$ in $L^p$ as wanted.

# $\overline{C^\infty(E) \cap L^p(E)} = L^p(E)$

Let $C^\infty(E) = \left\{ f|_E : f \in C^\infty(\mathbb{R}^n) \right\}$.

**Theorem**

*For $1 \leq p < \infty$, the space $C^\infty(E) \cap L^p(E)$ is dense in $L^p(E)$.*

Proof

- Fix $f \in L^p(E)$. We will produce functions $f_k \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $f_k|_E \to f$ in $L^p(E)$.
- Extend $f$ to $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $\tilde{f} = 0$ in $\mathbb{R}^n \setminus E$. Then $\tilde{f} \in L^p(\mathbb{R}^n)$. By the previous theorem, there exist $f_k \in C_c^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that $f_k \to \tilde{f}$ in $L^p(\mathbb{R}^n)$.
- Now,
$$\int_E |f_k - f|^p \, dx \leq \int_{\mathbb{R}^n} |f_k - \tilde{f}|^p \, dx \to 0,$$
and so $f_k|_E \to f$ in $L^p(E)$.

# Pre-compactness criterion in $C(K)$

## Theorem (Ascoli-Arzelà's theorem)

*Let $K$ be a compact subset of $\mathbb{R}^n$. Suppose that $(f_i)$ is a sequence of functions of $C(K)$ such that*

1. *(Boundedness)* $\sup_i \|f_i\|_{C(K)} < \infty$,
2. *(Equi-continuity) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for all $i$ and all $x, y \in K$ with $|x - y| < \delta$.*

*Then there exists a subsequence $(f_{i_j})$ which converges uniformly on $K$.*

In other words, the set $\{f_i\}$ is pre-compact.

# Pre-compactness criterion in $C(K)$

Proof

- We would like to show that $(f_i)$ has a subsequence $(f_{i_j})$ which is Cauchy in $C(K)$, i.e. for every given $\varepsilon > 0$,

$$\|f_{i_j} - f_{i_k}\|_{C(K)} \leq \varepsilon \text{ for all large } j, k. \tag{*}$$

- We claim that a slightly softer statement holds: For every given $\varepsilon$, there is a subsequence $(f^\varepsilon_{i_j})$ of $(f_i)$ such that

$$\|f^\varepsilon_{i_j} - f^\varepsilon_{i_k}\|_{C(K)} \leq 3\varepsilon \text{ for large } j, k. \tag{**}$$

- Suppose that (**) holds for the moment, we will now show how (*) can be obtained.

# Pre-compactness criterion in $C(K)$

Proof

- (**) $\Rightarrow$ (*): We will use a diagonal procedure.
  - Using (**), take a subsequence $(f^1_{i_j})$ of $(f_i)$ such that $\|f^1_{i_j} - f^1_{i_k}\|_{C(K)} \leq 1$ eventually.
  - Now the sequence $(f^1_{i_j})$ satisfies the condition of theorem. Since we are assuming (**), we can thus take a subsequence $(f^2_{i_j})$ of $(f^1_{i_j})$ such that $\|f^2_{i_j} - f^2_{i_k}\|_{C(K)} \leq 1/2$ eventually.
  - Proceeding inductively, we have a nest sequence of subsequences $(f_i) \supset (f^1_{i_j}) \supset (f^2_{i_j}) \supset \dots$ such that, for each $m \geq 1$,

$$\|f^m_{i_j} - f^m_{i_k}\|_{C(K)} \leq 1/m \text{ eventually.}$$

  - Now let $f_{i_j} = f^j_{i_j}$. Then, for every fixed $m$, the sequence $(f_{i_j})$ is eventually a subsequence of $(f^m_{i_j})$ and so $\|f_{i_j} - f_{i_k}\|_{C(K)} \leq 1/m$ eventually. So $(f_{i_j})$ satisfies (*).

# Pre-compactness criterion in $C(K)$

Proof

- We now prove (**), i.e. for every given $\varepsilon$, there is a subsequence $(f_{i_j}^\varepsilon)$ of $(f_i)$ such that

$$\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{C(K)} \leq 3\varepsilon \text{ for large } j, k.$$

  - ⋆ By equi-continuity, there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for all $i$ and all $x, y \in K$ with $|x - y| < \delta$.
  - ⋆ As $K$ is compact, we can cover $K$ by finitely many open balls $B(x_1, \delta), \ldots, B(x_N, \delta)$ with $x_\ell$'s in $K$.
  - ⋆ By uniform boundedness, for each $\ell$, the sequence $(f_i(x_\ell))$ is bounded in $\mathbb{R}$. By Bolzano-Weierstrass' theorem, we can select a subsequence $(f_{i_j}^\varepsilon)$ such that $(f_{i_j}^\varepsilon(x_\ell))$ is convergent for all $\ell$. So

$$|f_{i_j}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x_\ell)| \leq \varepsilon \text{ for all } \ell \text{ and for all large } j, k.$$

# Pre-compactness criterion in $C(K)$

Proof

- We now prove (**).
  - ⋆ ...$|f_i(x) - f_i(y)| \leq \varepsilon$ for all $i$ and all $x, y \in K$ with $|x - y| < \delta$.
  - ⋆ $B(x_1, \delta), \ldots, B(x_N, \delta)$ covers $K$.
  - ⋆ ...$|f_{i_j}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x_\ell)| \leq \varepsilon$ for all $\ell$ and for all large $j$ and $k$.
  - ⋆ Now if $x \in K$, then $x \in B(x_\ell, \delta)$ for some $\ell$. Then, for large $j, k$,

$$\begin{aligned} |f_{i_j}^\varepsilon(x) - f_{i_k}^\varepsilon(x)| &\leq |f_{i_j}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x_\ell)| + |f_{i_j}^\varepsilon(x_\ell) - f_{i_j}^\varepsilon(x)| \\ &\quad + |f_{i_k}^\varepsilon(x_\ell) - f_{i_k}^\varepsilon(x)| \\ &\leq 3\varepsilon. \end{aligned}$$

  - ⋆ So $\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{C(K)} \leq 3\varepsilon$, which proves (**).

# Pre-compactness criterion in $L^p(\Omega)$

## Theorem (Kolmogorov-Riesz-Fréchet's theorem)

*Let $1 \leq p < \infty$ and $\Omega$ be an open bounded subset of $\mathbb{R}^n$. Suppose that a sequence $(f_i)$ of $L^p(\Omega)$ satisfies*

1. *(Boundedness)* $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
2. *(Equi-continuity in $L^p$) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where $\tilde{f}_i$ is the extension by zero of $f_i$ to the whole of $\mathbb{R}^n$.*

*Then, there exists a subsequence $(f_{i_j})$ which converges in $L^p(\Omega)$.*

By definition $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{f}_i = f_i$ in $\Omega$ and $\tilde{f}_i = 0$ in $\mathbb{R}^n \setminus \Omega$.