Numerical Solution of Partial Differential Equations

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Lecture 1

Elements of function spaces

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We present a brief overview of definitions and basic results form the theory of function spaces, focusing in particular on spaces of:

- Continuous functions;
- Integrable functions; and
- Sobolev spaces.

Spaces of continuous functions

$\mathbb{N}$ denotes the set of nonnegative integers.
An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is called a *multi-index*.
The nonnegative integer $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is called the length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. We denote $(0, \ldots, 0)$ by $\mathbf{0}$; clearly $|\mathbf{0}| = 0$.

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Let

$$D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

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EXAMPLE. Suppose that $n = 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_j \in \mathbb{N}$, $j = 1, 2, 3$. Then, for u, a function of three variables x_1, x_2, x_3:

$$\sum_{|\alpha|=3} D^\alpha u = \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1^2 \partial x_3} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^3}$$
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We shall frequently write ∂_{x_j} instead of $\frac{\partial}{\partial x_j}$. ◇

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We denote by $C^k(\Omega)$ the set of all continuous real-valued functions defined on Ω s.t. $D^\alpha u$ is continuous on Ω for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

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Assuming that Ω is a *bounded* open set, $C^k(\overline{\Omega})$ will denote the set of all u in $C^k(\Omega)$ s.t. $D^\alpha u$ can be extended from Ω to a continuous function on $\overline{\Omega}$, the closure of the set Ω, for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

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The linear space $C^k(\overline{\Omega})$ can then be equipped with the norm

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \le k} \sup_{x \in \Omega} \left| D^\alpha u(x) \right| .$$

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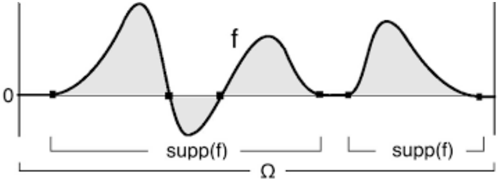
Note: When $k = 0$, we shall write $C(\overline{\Omega})$ instead of $C^0(\overline{\Omega})$.

The *support*, supp f, of a continuous function f on Ω is defined as the closure in Ω of the set

$$\{x \in \Omega : f(x) \neq 0\}.$$

In other words, supp f is the smallest closed subset of Ω such that $f = 0$ in $\Omega\backslash$supp u.



EXAMPLE. Let f be the function defined on $\mathbb{R}^n$ by

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & \text{otherwise}; \end{cases}$$

here $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$ for $x \in \mathbb{R}^n$.

Clearly, supp f is the closed unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$. ◇

We denote by $C_0^k(\Omega)$ the set of all $u \in C^k(\Omega)$ such that supp $u \subset \Omega$ and supp u is bounded. Let

$$C_0^\infty(\Omega) = \bigcap_{k \geq 0} C_0^k(\Omega).$$

EXAMPLE.

The function f defined in the previous example belongs to $C_0^\infty(\mathbb{R}^n)$. $\diamond$

Spaces of integrable functions

Let p be a real number, $p \geq 1$; we denote by $L_p(\Omega)$ the set of all real-valued functions defined on Ω such that

$$\int_{\Omega} \left| u(x) \right|^{p} \mathrm{d}x < \infty.$$

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Functions which are equal almost everywhere (i.e., equal, except on a set of measure zero) on Ω are identified with each other.

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A subset of $\mathbb{R}^n$ is said to be a *set of measure zero* if it can be contained in the union of countably many open balls of arbitrarily small total volume.

$L_p(\Omega)$ is equipped with the norm

$$\|u\|_{L_p(\Omega)} := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

A particularly important case is $p = 2$; then,

$$\|u\|_{L_2(\Omega)} = \left(\int_\Omega \big|u(x)\big|^2 \,\mathrm{d}x \right)^{1/2}.$$

The space $L_2(\Omega)$ can be equipped with an inner product

$$(u, v) := \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

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Lemma (The Cauchy–Schwarz inequality)

Let $u, v \in L_2(\Omega)$; then

$$\left|(u, v)\right| \leq \|u\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.$$

Remark. The space $L_2(\Omega)$ equipped with the inner product $(\cdot,\cdot)$ (and the associated norm $\|u\|_{L_2(\Omega)} = (u,u)^{1/2}$) is an example of a Hilbert space.

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In general, a linear space X, equipped with an inner product $(\cdot, \cdot)_X$ (and the associated norm $\|u\|_X = (u, u)_X^{1/2}$) is called a Hilbert space if, whenever $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in X, i.e. a sequence of elements of X such that

$$\lim_{n,\, m \to \infty} \|u_n - u_m\|_X = 0,$$

then there exists a $u \in X$ such that $\lim_{m \to \infty} \|u - u_m\|_X = 0$ (i.e., the sequence $\{u_m\}_{m=1}^{\infty}$ converges to u in the norm of X).

Sobolev spaces

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$$\int_\Omega w_\alpha(x)\, v(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x) \quad \forall\, v \in C_0^\infty(\Omega).$$

Then w_α is called the *weak derivative* of u (of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$) and we write $w_\alpha = D^\alpha u$.

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Clearly, if u is a smooth function then its weak derivatives coincide with those in the classical (pointwise) sense. To simplify the notation, we shall use the letter D to denote both a classical and a weak derivative.

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EXAMPLE Let $\Omega = \mathbb{R}^1$, and let $u(x) = (1 - |x|)_+$. Here, for a real number y, $y_+ := \max\{y, 0\}$. Clearly u is not differentiable at $x = 0, \pm 1$. However, because u is locally integrable on Ω it may still have a weak derivative. Indeed, for any $v \in C_0^\infty(\Omega)$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} u(x)\, v'(x)\, \mathrm{d}x &= \int_{-\infty}^{+\infty} (1 - |x|)_+\, v'(x)\, \mathrm{d}x = \int_{-1}^{1} (1 - |x|)\, v'(x)\, \mathrm{d}x \\
&= \int_{-1}^{0} (1 + x)\, v'(x)\, \mathrm{d}x + \int_{0}^{1} (1 - x)\, v'(x)\, \mathrm{d}x \\
&= \int_{-1}^{0} (-1)\, v(x)\, \mathrm{d}x + \int_{0}^{1} (+1)\, v(x)\, \mathrm{d}x \\
&= -\int_{-\infty}^{+\infty} w(x)\, v(x)\, \mathrm{d}x,
\end{aligned}$$

where

$$w(x) = \begin{cases} 0, & x < -1, \\ 1, & x \in (-1, 0), \\ -1, & x \in (0, 1), \\ 0, & x > 1. \end{cases}$$

Thus, $w = u' = Du$ $\diamond$

Let k be a nonnegative integer. We define (with D^α denoting a weak derivative of order $|\alpha|$)

$$H^k(\Omega) := \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), \quad |\alpha| \leq k\}.$$

$H^k(\Omega)$ is called a Sobolev space of order k; it is equipped with the (Sobolev) norm

$$\|u\|_{H^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{1/2}$$

and the inner product

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v).$$

With this inner product, $H^k(\Omega)$ is a Hilbert space (for the definition of Hilbert space, see the remark in Section 1.2).

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$|\cdot|_{H^k(\Omega)}$ is called the Sobolev semi-norm (it is only a semi-norm rather than a norm because if $|u|_{H^k(\Omega)} = 0$ for $u \in H^k(\Omega)$ it does not necessarily follow that $u \equiv 0$ on Ω.)

EXAMPLE

$$H^0(\Omega) = L_2(\Omega).$$

$$H^1(\Omega) := \left\{ u \in L_2(\Omega) : \partial_{x_j} u := \frac{\partial u}{\partial x_j} \in L_2(\Omega), \ \ j = 1, \dots, n \right\},$$

$$\|u\|_{H^1(\Omega)} := \left\{ \|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L_2(\Omega)} \right\}^{1/2},$$

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Similarly,

$$H^2(\Omega) := \left\{ u \in L_2(\Omega) : \partial_{x_j} u \in L_2(\Omega),\ \partial^2_{x_i x_j} u \in L_2(\Omega),\ i,j = 1,\ldots,n \right\},$$

$$\|u\|_{H^2(\Omega)} := \left\{ \|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L_2(\Omega)} + \sum_{i,j=1}^{n} \|\partial^2_{x_i x_j} u\|^2_{L_2(\Omega)} \right\}^{1/2},$$

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We define a special Sobolev space,

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

i.e. $H_0^1(\Omega)$ is the set of all functions u in $H^1(\Omega)$ such that $u = 0$ on $\partial\Omega$, the boundary of the set Ω.

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$H_0^1(\Omega)$ is a Hilbert space, with the same norm and inner product as $H^1(\Omega)$.

We conclude with the following important result.

Lemma (Poincaré–Friedrichs inequality)

Suppose that Ω is a bounded open set in $\mathbb{R}^n$ (with a sufficiently smooth boundary $\partial\Omega$) and let $u \in H_0^1(\Omega)$; then, there exists a positive constant $c_\star(\Omega)$, independent of u, such that

$$\int_\Omega u^2(x)\,\mathrm{d}x \leq c_\star \sum_{i=1}^{n} \int_\Omega \left|\partial_{x_i} u(x)\right|^2 \,\mathrm{d}x. \tag{1}$$

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$$u(x, y) = u(a, y) + \int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi = \int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi, \qquad c < y < d.$$

Proof. We shall prove this for the special case of a rectangular domain $\Omega = (a, b) \times (c, d)$ in $\mathbb{R}^2$. The proof for general Ω is analogous. Evidently,

$$u(x, y) = u(a, y) + \int_a^x \partial_x u(\xi, y)\, \mathrm{d}\xi = \int_a^x \partial_x u(\xi, y)\, \mathrm{d}\xi, \qquad c < y < d.$$

Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_\Omega \left|u(x, y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_c^d \left|\int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi\right|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_a^b \int_c^d (x - a) \left(\int_a^x \left|\partial_x u(\xi, y)\right|^2 \,\mathrm{d}\xi\right) \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_a^b (x - a)\,\mathrm{d}x \left(\int_c^d \int_a^b \left|\partial_x u(\xi, y)\right|^2 \,\mathrm{d}\xi\,\mathrm{d}y\right) \\
&= \frac{1}{2}(b - a)^2 \int_\Omega \left|\partial_x u(x, y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Analogously,

$$\int_\Omega \left|u(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq \frac{1}{2}(d-c)^2 \int_\Omega \left|\partial_y u(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Analogously,

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By adding the two inequalities, we obtain

$$\int_\Omega \left|u(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \le c_\star \int_\Omega \left(\left|\partial_x u\right|^2 + \left|\partial_y u\right|^2\right) \,\mathrm{d}x\,\mathrm{d}y,$$

where $c_\star = \left(\dfrac{2}{(b-a)^2} + \dfrac{2}{(d-c)^2}\right)^{-1}.$ □