# Numerical Solution of Partial Differential Equations


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Lecture 12

# Finite difference approximation of hyperbolic equations

In the remaining lectures we shall discuss the finite difference approximation of hyperbolic PDEs. In this lecture, and in Lectures 13 and 14 we shall focus on second-order hyperbolic PDEs, and in Lectures 15 and 16 on first-order hyperbolic PDEs.

The simplest example of a second-order linear hyperbolic equation is the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t),$$

where $c > 0$ is the wave speed and $f$ is a given source term.

When $f$ is identically zero and the equation is considered on the whole real line, $x \in \mathbb{R}$, by supplying two initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for } x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(x,0) &= u_1(x) \quad \text{for } x \in \mathbb{R}, \end{aligned}$$

where $u_0$ and $u_1$ are defined on $\mathbb{R}$, $u_0$ is twice continuously differentiable and $u_1$ is once continuously differentiable on $\mathbb{R}$, the solution is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,\mathrm{d}\xi.$$

More generally, if $f$ is a continuous function on $\mathbb{R} \times [0, \infty)$, there is still an explicit formula for the solution:

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,\mathrm{d}\xi \quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau.$$

We shall be interested in a problem of the above form, but in the physically more realistic setting where $x$ is confined to a nonempty bounded closed spatial interval $[a, b]$, with $a < b$, and where $t \in [0, T]$, with $T > 0$.

Then, in addition to the two initial conditions above, boundary conditions need to be prescribed at $x = a$ and $x = b$, and the problem thus becomes an initial-boundary-value problem.

Consider the closed interval $[a, b]$ of the real line, with $a < b$, and let $T > 0$. We shall be concerned with the finite difference approximation of the initial-boundary-value problem

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= f(x, t) && \text{for } (x, t) \in (a, b) \times (0, T], \\
u(x, 0) &= u_0(x) && \text{for } x \in [a, b], \\
\frac{\partial u}{\partial t}(x, 0) &= u_1(x) && \text{for } x \in [a, b], \\
u(a, t) = 0 \quad \text{and} \quad u(b, t) &= 0 && \text{for } t \in [0, T].
\end{aligned}
\tag{1}
$$

Here, $f$ is assumed to be a continuous real-valued function defined on $(a, b) \times [0, T]$, $u_0$ and $u_1$ are supposed to be continuous real-valued functions defined on $[a, b]$, and we shall assume compatibility of the initial data with the boundary conditions, in the sense that $u_0$ and $u_1$ are required to vanish at both $x = a$ and $x = b$. As before, $c > 0$ is the wave speed.

The analysis of the finite difference approximation of (1) is based on 'discrete energy inequalities', which will imply the stability of the finite difference schemes under consideration, and which will also play a key role in their convergence analysis.

We begin by describing the derivation of the 'energy inequality' (or 'energy estimate') for the solution of the initial-boundary-value problem (1).

The proof of existence of a solution to the initial-boundary-value problem (1) is beyond the scope of this course; we shall simply suppose here that a solution $u$ to (1) exists and that $u$ is sufficiently smooth, so that our calculations are meaningful.

We begin by multiplying the PDE $(1)_1$ by the time derivative of $u$, and we then integrate over the interval $[a, b]$; thus,

$$\begin{aligned} \int_a^b \frac{\partial^2 u}{\partial t^2}(x,t)\,\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x - c^2 \int_a^b \frac{\partial^2 u}{\partial x^2}(x,t)\,\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x \\ = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x. \end{aligned} \tag{2}$$

As $u(a,t) = 0$ and $u(b,t) = 0$ for all $t \in [0, T]$, it follows that

$$\frac{\partial u}{\partial t}(a,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(b,t) = 0 \quad \text{for all } t \in [0, T].$$

By performing partial integration with respect to $x$ in the second term on the left-hand side of (2), we have that

$$\begin{aligned}\int_a^b \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}(x,t)\right)\,\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x \\ + c^2\int_a^b \frac{\partial u}{\partial x}(x,t)\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}(x,t)\right)\,\mathrm{d}x = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x.\end{aligned} \tag{3}$$

Clearly,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\,\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2,$$

and therefore

$$\begin{aligned}\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b\left(\frac{\partial u}{\partial t}\right)^2(x,t)\,\mathrm{d}x + \frac{c^2}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b\left(\frac{\partial u}{\partial x}\right)^2(x,t)\,\mathrm{d}x \\ = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x.\end{aligned} \tag{4}$$

When $f$ is identically zero, the r.h.s. of (4) vanishes, and after integrating the expression from 0 to $t$, for any $t \in (0, T]$, we have

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + \frac{c^2}{2}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,0)\,\mathrm{d}x + \frac{c^2}{2}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,0)\,\mathrm{d}x.
\end{aligned} \tag{5}$$

The l.h.s. side of the equality (5) can be viewed as the 'total energy' at time $t$ and the r.h.s. as the 'initial total energy'. Thus, (5) expresses conservation of the total energy during the course of the evolution of the solution from time 0 to time $t \in (0, T]$, in the absence of a source term.

After multiplying (4) by 2 and defining

$$\mathcal{L}^2(u(\cdot,t)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x$$

for $t \in [0, T]$, the equality (4) can be rewritten as

$$\mathcal{L}^2(u(\cdot,t)) = \mathcal{L}^2(u(\cdot,0)) \qquad \text{for all } t \in [0, T].$$

It is this argument that we shall try to mimic in our stability analysis of the finite difference approximation of (4) when $f \equiv 0$.

We note that the mapping

$$u \mapsto \max_{t\in[0,T]} [\mathcal{L}^2(u(\cdot,t))]^{1/2}$$

is a norm on the linear space of continuous functions $u$ defined on $[a, b] \times [0, T]$ such that $u(a, t) = u(b, t) = 0$ for all $t \in [0, T]$, and whose first partial derivatives with respect to $x$ and $t$ are continuous functions defined on $[a, b] \times [0, T]$.

More generally, if $f$ is not identically zero, then (4) implies that

$$\mathcal{L}^2(u(\cdot,t)) = \mathcal{L}^2(u(\cdot,0)) + 2\int_0^t \int_a^b f(x,\tau)\frac{\partial u}{\partial t}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

As

$$2\alpha\beta \leq \alpha^2 + \beta^2, \qquad \text{for all } \alpha,\beta\in\mathbb{R},$$

it follows that

$$\begin{aligned}\mathcal{L}^2(u(\cdot,t)) &\leq \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \mathcal{L}^2(u(\cdot,\tau))\,\mathrm{d}\tau.\end{aligned} \tag{6}$$

To proceed, we need the following result.

### Lemma (Gronwall's Lemma)

*Suppose that $A$ and $B$ are continuous real-valued nonnegative functions defined on $[0, T]$, and $B$ is a nondecreasing function of its argument. Suppose further that*

$$A(t) \leq B(t) + \int_0^t A(s)\,\mathrm{d}s$$

*for all $t \in [0, T]$; then*

$$A(t) \leq \mathrm{e}^t\, B(t)$$

*for all $t \in [0, T]$.*

PROOF: Clearly,

$$\mathrm{e}^{-t} A(t) - \mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s \le \mathrm{e}^{-t} B(t),$$

and thus, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s\right] \le \mathrm{e}^{-t} B(t).$$

Hence, by integrating and noting that the expression in the square brackets on the l.h.s. of the last inequality vanishes at $t = 0$,

$$\mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s \le \int_0^t \mathrm{e}^{-s} B(s)\,\mathrm{d}s.$$

Multiplying this by $\mathrm{e}^t$, and since $B$ is a nondecreasing nonnegative function, whereby $B(s) \le B(t)$ for all $s \in [0, t]$, we have that

$$\int_0^t A(s)\,\mathrm{d}s \le \mathrm{e}^t B(t) \int_0^t \mathrm{e}^{-s}\,\mathrm{d}s = \mathrm{e}^t B(t)\,(1 - \mathrm{e}^{-t}) = \mathrm{e}^t B(t) - B(t).$$

Substituting this into the r.h.s. of the inequality in the statement of the lemma: $A(t) \le B(t) + \mathrm{e}^t B(t) - B(t) = \mathrm{e}^t B(t)$. $\square$

We now return to (6) and set

$$A(t) := \mathcal{L}^2(u(\cdot,t)) \quad \text{and} \quad B(t) := \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$

It then follows from Gronwall's lemma that $A(t) \le \mathrm{e}^t B(t)$, that is

$$\mathcal{L}^2(u(\cdot,t)) \le \mathrm{e}^t \left( \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right),$$

with

$$\mathcal{L}^2(u(\cdot,t)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x$$

and

$$\mathcal{L}^2(u(\cdot,0)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,0)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,0)\,\mathrm{d}x$$
$$= \|u_1\|^2_{L_2((a,b))} + c^2 \, |u_0|^2_{H^1((a,b))}$$

This is the desired energy inequality satisfied by the solution.

It provides a bound on the (square of the) norm of the solution in terms of the (square of the) norm of the initial data and the (square of the) $L_2$ norm of the source term $f$.

We shall mimic the derivation of this energy inequality in the stability analysis of the implicit and explicit finite difference approximations of the initial-boundary-value problem (1) in the general case when $f$ is not identically zero.