Numerical Solution of Partial Differential Equations

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Lecture 13

The implicit scheme: stability, consistency and convergence

Consider the closed interval $[a, b]$ of the real line, with $a < b$, and let $T > 0$. We shall be concerned with the finite difference approximation of the initial-boundary-value problem

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= f(x,t) && \text{for } (x,t) \in (a,b) \times (0,T], \\
u(x,0) &= u_0(x) && \text{for } x \in [a,b], \\
\frac{\partial u}{\partial t}(x,0) &= u_1(x) && \text{for } x \in [a,b], \\
u(a,t) = 0 \quad \text{and} \quad u(b,t) &= 0 && \text{for } t \in [0,T].
\end{aligned}
\tag{1}
$$

Here, f is assumed to be a continuous real-valued function defined on $(a, b) \times [0, T]$, u_0 and u_1 are supposed to be continuous real-valued functions defined on $[a, b]$, and we shall assume compatibility of the initial data with the boundary conditions, in the sense that u_0 and u_1 are required to vanish at both $x = a$ and $x = b$. As before, $c > 0$ is the wave speed.

For $M \geq 2$, we define $\Delta t := T/M$, and for $J \geq 2$ the spatial step is taken to be $\Delta x := (b - a)/J$. We let $x_j := a + j\Delta x$ for $j = 0, 1, \ldots, J$ and $t_m := m\Delta t$ for $m = 0, 1, \ldots, M$.

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On the space-time mesh $\{(x_j, t_m) : 0 \leq j \leq J,\ 0 \leq m \leq M\}$ we consider the finite difference scheme

$$
\begin{aligned}
\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{\Delta x^2} &= f(x_j, t_{m+1}) \quad \text{for} \begin{cases} j = 1, \ldots, J-1, \\ m = 1, \ldots, M-1, \end{cases} \\
U_j^0 &= u_0(x_j) \qquad \text{for } j = 0, 1, \ldots, J, \\
U_j^1 &= U_j^0 + \Delta t\, u_1(x_j) \quad \text{for } j = 1, 2, \ldots, J-1, \\
U_0^m = 0 \quad \text{and} \quad U_J^m &= 0 \qquad \text{for } m = 1, \ldots, M.
\end{aligned} \tag{2}
$$

The second numerical initial condition, featuring in equation $(2)_3$, stems from the observation that if $\frac{\partial^2 u}{\partial t^2} \in C([a, b] \times [0, T])$ then

$$\begin{aligned}\frac{u(x_j, \Delta t) - U_j^0}{\Delta t} &= \frac{u(x_j, \Delta t) - u(x_j, 0)}{\Delta t} \\ &= \frac{\partial u}{\partial t}(x_j, 0) + \mathcal{O}(\Delta t) = u_1(x_j) + \mathcal{O}(\Delta t);\end{aligned}$$

thus, by ignoring the $\mathcal{O}(\Delta t)$ term and replacing $u(x_j, \Delta t)$ by its numerical approximation U_j^1 we obtain $(2)_3$.

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Once the values of U_j^{m-1} and U_j^m, for $j = 0,\ldots,J$, have been computed (or have been specified by the initial data, in the case of $m = 1$), the subsequent values U_j^{m+1}, $j = 0,\ldots,J$, need to be computed by solving a system of $J-1$ linear algebraic equations for the $J-1$ unknowns U_j^{m+1}, $j = 0,\ldots,J-1$, for $m = 0,\ldots,M-1$. The finite difference scheme (2) is therefore called the *implicit scheme* for the initial-boundary-value problem.

Stability of the implicit scheme.

Consider the inner products

$$(U, V) := \sum_{j=1}^{J-1} \Delta x \, U_j \, V_j,$$

$$(U, V] := \sum_{j=1}^{J} \Delta x \, U_j \, V_j,$$

and the associated norms, respectively, $\| \cdot \|$ and $\| \cdot]|$, defined by

$$\|U\| := (U, U)^{\frac{1}{2}} \quad \text{and} \quad \|U]| := (U, U]^{\frac{1}{2}}.$$

Note that for two mesh functions A and B defined on the computational mesh $\{x_j : j = 1, \dots, J-1\}$ one has that (please check this!)

$$(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A - B\|^2.$$

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Thus, by taking $A = U^{m+1} - U^m$ and $B = U^m - U^{m-1}$, we have

$$\begin{aligned}
&(U^{m+1} - 2U^m + U^{m-1}, U^{m+1} - U^m) \\
&= \frac{1}{2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{1}{2}\|U^{m+1} - 2U^m + U^{m-1}\|^2.
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$$\begin{aligned}&(U^{m+1} - 2U^m + U^{m-1}, U^{m+1} - U^m)\\ &= \frac{1}{2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{1}{2}\|U^{m+1} - 2U^m + U^{m-1}\|^2.\end{aligned}$$

Similarly as above, for two mesh functions A and B defined on the computational mesh $\{x_j : j = 1, \ldots, J\}$ we have that

$$(A - B, A] = \frac{1}{2}(\|A]|^2 - \|B]|^2) + \frac{1}{2}\|A - B]|^2.$$

Hence, by performing a summation by parts and then taking $A = D_x^+ U^{m+1}$ and $B = D_x^+ U^m$ we have

$$\begin{aligned}\left(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m\right) &= (D_x^- U^{m+1}, D_x^-(U^{m+1} - U^m)] \\ &= (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1}] \\ &= \frac{1}{2}(\|D_x^- U^{m+1}]|^2 - \|D_x^- U^m]|^2) \\ &\quad + \frac{1}{2}\|D_x^-(U^{m+1} - U^m)]|^2.\end{aligned}$$

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$$\begin{aligned}\left(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m\right) &= (D_x^- U^{m+1}, D_x^-(U^{m+1} - U^m)] \\ &= (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1}] \\ &= \frac{1}{2}(\|D_x^- U^{m+1}]|^2 - \|D_x^- U^m]|^2) \\ &\quad + \frac{1}{2}\|D_x^-(U^{m+1} - U^m)]|^2.\end{aligned}$$

By taking the $(\cdot,\cdot)$ inner product of $(2)_1$ with $U^{m+1} - U^m$ and using the identities stated above we therefore obtain:

$$\begin{aligned}&\frac{1}{2}\left(\left\|\frac{U^{m+1} - U^m}{\Delta t}\right\|^2 - \left\|\frac{U^m - U^{m-1}}{\Delta t}\right\|^2\right) + \frac{1}{2}\Delta t^2\left\|\frac{U^{m+1} - 2U^m + U^{m-1}}{\Delta t^2}\right\|^2 \\ &\quad + \frac{1}{2}c^2(\|D_x^- U^{m+1}]|^2 - \|D_x^- U^m]|^2) + \frac{1}{2}c^2\Delta t^2\left\|D_x^-\left(\frac{U^{m+1} - U^m}{\Delta t}\right)\right]\Bigg|^2 \\ &\quad = (f(\cdot, t_{m+1}), U^{m+1} - U^m).\end{aligned} \tag{3}$$

In the special case when f is identically zero the equality (3) gives

$$\left\|\frac{U^{m+1}-U^{m}}{\Delta t}\right\|^2 + c^2\|D_x^- U^{m+1}]|^2 \leq \left\|\frac{U^{m}-U^{m-1}}{\Delta t}\right\|^2 + c^2\|D_x^- U^{m}]|^2. \quad (4)$$

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Let us define the nonnegative expression

$$\mathcal{M}^2(U^m) := \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2 + c^2 \| D_x^- U^{m+1}]|^2.$$

With this notation (4) becomes

$$\mathcal{M}^2(U^m) \le \mathcal{M}^2(U^{m-1}), \qquad \text{for all } m = 1, \dots, M-1,$$

and therefore

$$\mathcal{M}^2(U^m) \le \mathcal{M}^2(U^0), \qquad \text{for all } m = 1, \dots, M-1.$$

One can verify (please check this!) that the mapping

$$U \mapsto \max_{m \in \{0,\dots,M-1\}} [\mathcal{M}^2(U^m)]^{1/2}$$

is a norm on the linear space of mesh functions U defined on the space-time mesh $\{(x_j, t_m) : j = 0, 1, \dots, J,\ m = 0, 1, \dots, M\}$ such that $U_0^m = U_J^m = 0$ for all $m = 0, 1, \dots, M$.

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Thus we have shown that when f is identically zero the implicit scheme (2) is (unconditionally) stable in this norm.

Note: Unfortunately, the implicit scheme only satisfies an *energy inequality* rather than an energy equality when $f \equiv 0$.

We now return to the general case when f is not identically zero. Our starting point is the equality (3) and we focus our attention on the term on its right-hand side. By the Cauchy–Schwarz inequality,

$$\begin{aligned}(f(\cdot, t_{m+1}), U^{m+1} - U^m) &\leq \|f(\cdot, t_{m+1})\| \, \|U^{m+1} - U^m\| \\ &= \sqrt{\Delta t\, T}\, \|f(\cdot, t_{m+1})\| \sqrt{\frac{\Delta t}{T}} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\| \\ &\leq \frac{\Delta t\, T}{2} \|f(\cdot, t_{m+1})\|^2 + \frac{\Delta t}{2T} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2, \end{aligned} \tag{5}$$

where in the transition to the last line we used the elementary inequality

$$\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2, \qquad \text{for } \alpha, \beta \in \mathbb{R}.$$

Substituting (5) into (3) we deduce that

$$\left(1-\frac{\Delta t}{T}\right)\left(\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|^2+c^2\|D_x^- U^{m+1}]|^2\right) \\ \leq \left\|\frac{U^m-U^{m-1}}{\Delta t}\right\|^2+c^2\|D_x^- U^m]|^2+\Delta t\, T\,\|f(\cdot,t_{m+1})\|^2. \tag{6}$$

Substituting (5) into (3) we deduce that

$$\left(1-\frac{\Delta t}{T}\right)\left(\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|^2 + c^2\|D_x^- U^{m+1}]|^2\right) \le \left\|\frac{U^m-U^{m-1}}{\Delta t}\right\|^2 + c^2\|D_x^- U^m]|^2 + \Delta t\, T\, \|f(\cdot, t_{m+1})\|^2. \tag{6}$$

By recalling the definition of $\mathcal{M}^2(U^m)$ we can rewrite (6) in the following compact form:

$$\left(1-\frac{\Delta t}{T}\right)\mathcal{M}^2(U^m) \le \mathcal{M}^2(U^{m-1}) + \Delta t\, T\, \|f(\cdot, t_{m+1})\|^2$$

for $m = 1, 2, \ldots, M-1$.

As, by assumption, $M \geq 2$, it follows that $\Delta t := T/M \leq T/2$, whereby $\Delta t/T \leq 1/2$. By noting that

$$1 - x \geq \frac{1}{1+2x} \qquad \forall\, x \in \left[0, \tfrac{1}{2}\right],$$

it follows with $x = \Delta t/T$ that

$$\begin{aligned} \mathcal{M}^2(U^m) &\leq \left(1 + \frac{2\,\Delta t}{T}\right) \mathcal{M}^2(U^{m-1}) + \Delta t\; T \left(1 + \frac{2\,\Delta t}{T}\right) \|f(\cdot, t_{m+1})\|^2 \\ &\leq \left(1 + \frac{2\,\Delta t}{T}\right) \mathcal{M}^2(U^{m-1}) + 2\,\Delta t\; T\, \|f(\cdot, t_{m+1})\|^2 \end{aligned}$$

for $m = 1, 2, \ldots, M-1$.

We need the following result, which is easily proved by induction.

Lemma

Suppose that $M \geq 2$ *is an integer,* $\{a_m\}_{m=0}^{M-1}$ *and* $\{b_m\}_{m=1}^{M-1}$ *are nonnegative real numbers,* $\alpha > 0$*, and*

$$a_m \leq \alpha\, a_{m-1} + b_m \qquad \textit{for } m = 1, 2, \ldots, M-1.$$

Then,

$$a_m \leq \alpha^m a_0 + \sum_{k=1}^{m} \alpha^{m-k} b_k \qquad \textit{for } m = 1, 2, \ldots, M-1.$$

We shall apply Lemma 1 with

$$a_m = \mathcal{M}^2(U^m), \quad b_m = 2\,\Delta t\; T\,\|f(\cdot, t_{m+1})\|^2, \quad \alpha = 1 + \frac{2\,\Delta t}{T}$$

to deduce that, for $m = 1, 2, \ldots, M-1$,

$$\mathcal{M}^2(U^m) \le \left(1 + \frac{2\,\Delta t}{T}\right)^m \mathcal{M}^2(U^0) + 2\,\Delta t\; T \sum_{k=1}^{m} \left(1 + \frac{2\,\Delta t}{T}\right)^{m-k} \|f(\cdot, t^{k+1})\|^2.$$

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We note that

$$\left(1 + \frac{2\,\Delta t}{T}\right)^m \leq \left(1 + \frac{2\,\Delta t}{T}\right)^M = \left(1 + \frac{2\,\Delta t}{T}\right)^{\frac{T}{\Delta t}} \leq \mathrm{e}^2,$$

where the last inequality follows from the inequality

$$(1 + 2x)^{\frac{1}{x}} \leq e^2 \quad \forall\, x \in \left(0, \tfrac{1}{2}\right],$$

with $x = \Delta t/T$, which follows by noting that $\log(1 + x) \leq x$ for all $x \geq 0$.

Thus we deduce the following stability result for the implicit scheme (2).

Theorem

The implicit finite difference approximation (2) *of the initial-boundary-value problem, on a finite difference mesh of spacing* $\Delta x = (b-a)/J$ *with* $J \geq 2$ *in the* x*-direction and* $\Delta t = T/M$ *with* $M \geq 2$ *in the* t*-direction, is (unconditionally) stable in the sense that, for* $m = 1, \ldots, M-1$,

$$\mathcal{M}^2(U^m) \leq \mathrm{e}^2\,\mathcal{M}^2(U^0) + 2\,\mathrm{e}^2\,T \sum_{k=1}^{m} \Delta t \, \|f(\cdot, t_{k+1})\|^2 \,,$$

independently of the choice of Δx *and* Δt.

Consistency of the implicit scheme.

We define the consistency error of the scheme for $m = 1, \ldots, M-1$ by

$$T_j^{m+1} := \frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{\Delta t^2} - c^2 \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2} - f(x_j, t_{m+1}),$$

and

$$T_j^1 := \frac{u_j^1 - u_j^0}{\Delta t} - u_1(x_j), \qquad j = 1, \ldots, J-1,$$

where $u_j^m := u(x_j, t_m)$.

As

$$f(x_j, t_{m+1}) = \frac{\partial^2 u}{\partial t^2}(x_j, t_{m+1}) - c^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_{m+1}) \quad \text{and} \quad u_1(x_j) = \frac{\partial u}{\partial t}(x_j, 0),$$

it follows that

$$\begin{aligned} T_j^{m+1} := & \left(\frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{\Delta t^2} - \frac{\partial^2 u}{\partial t^2}(x_j, t_{m+1}) \right) \\ & - c^2 \left(\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2}(x_j, t_{m+1}) \right) \end{aligned}$$

for $j = 1, \ldots, J-1$ and $m = 1, \ldots, M-1$ and

$$T_j^1 = \frac{u_j^1 - u_j^0}{\Delta t} - \frac{\partial u}{\partial t}(x_j, 0)$$

for $j = 1, \ldots, J-1$.

By Taylor series expansion of $u_j^m = u(x_j, t_m)$ and $u_j^{m-1} = u(x_j, t_{m-1})$ about the point (x_j, t_{m+1}) we have that

$$\frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{\Delta t^2} - \frac{\partial^2 u}{\partial t^2}(x_j, t_{m+1}) = \frac{1}{3}\Delta t \left(\frac{\partial^3 u}{\partial t^3}(x_j, \eta_m) - 4\frac{\partial^3 u}{\partial t^3}(x_j, \eta_{m-1}) \right),$$

where $\eta_{m-1} \in [t_{m-1}, t_{m+1}]$ and $\eta_m \in [t_m, t_{m+1}]$, provided that the third partial derivative of u w.r.t. t is a continuous function on $[a, b] \times [0, T]$.

Similarly, by Taylor series expansion of $u_{j+1}^{m+1} = u(x_{j+1}, t_{m+1})$ and $u_{j-1}^{m+1} = u(x_{j-1}, t_{m+1})$ about the point (x_j, t_{m+1}) we find that

$$\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2}(x_j, t_{m+1}) = \frac{1}{12}\Delta x^2 \frac{\partial^4 u}{\partial x^4}(\xi_j, t_{m+1}),$$

where $\xi_j \in [x_{j-1}, x_{j+1}]$, provided that the fourth partial derivative of u with respect to x is a continuous function on $[a, b] \times [0, T]$.

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where $\xi_j \in [x_{j-1}, x_{j+1}]$, provided that the fourth partial derivative of u with respect to x is a continuous function on $[a, b] \times [0, T]$. Hence,

$$|T_j^{m+1}| \le \frac{1}{12}c^2\Delta x^2 M_{4x} + \frac{5}{3}\Delta t M_{3t}, \qquad \begin{cases} j = 1, \ldots, J-1, \\ m = 1, \ldots, M-1, \end{cases} \tag{7}$$

where

$$M_{4x} := \max_{(x,t)\in[a,b]\times[0,T]} \left|\frac{\partial^4 u}{\partial x^4}(x, t)\right| \quad \text{and} \quad M_{3t} := \max_{(x,t)\in[a,b]\times[0,T]} \left|\frac{\partial^3 u}{\partial t^3}(x, t)\right|.$$

It remains to bound T_j^1. This time, by performing a Taylor series expansion, but now with an integral remainder term, we get that

$$T_j^1 = \frac{1}{\Delta t} \int_0^{\Delta t} (\Delta t - t) \, \frac{\partial^2 u}{\partial t^2}(x_j, t) \, \mathrm{d}t, \tag{8}$$

and therefore

$$|\, T_j^1| \leq \frac{1}{2} \Delta t \, M_{2t}, \quad j = 1, \ldots, J-1,$$

where

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Having bounded the consistency error we are now ready to investigate the convergence of the implicit scheme.

Convergence of the implicit scheme

In the rest of the section we shall explore the convergence of the finite difference scheme (2). To this end, we define the *global error*

$$e_j^m := u(x_j, t_m) - U_j^m, \qquad \begin{cases} j = 0, \ldots, J, \\ m = 0, \ldots, M. \end{cases}$$

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It follows from the definitions of T_j^{m+1} and T_j^1 that

$$\frac{e_j^{m+1} - 2e_j^m + e_j^{m-1}}{\Delta t^2} - c^2 \frac{e_{j+1}^{m+1} - 2e_j^{m+1} + e_{j-1}^{m+1}}{\Delta x^2} = T_j^{m+1},$$

for $j = 1, \dots, J-1$ and $m = 1, \dots, M-1$, and

$$e_j^1 = e_j^0 + \Delta t\, T_j^1, \qquad j = 1, \dots, J-1.$$

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for $j = 1, \ldots, J-1$ and $m = 1, \ldots, M-1$, and

$$e_j^1 = e_j^0 + \Delta t \, T_j^1, \qquad j = 1, \ldots, J-1.$$

Furthermore, $e_j^0 = 0$ for $j = 0, 1, \ldots, J$, and $e_0^m = e_J^m = 0$ for $m = 1, \ldots, M$.

Hence, the global error e satisfies an identical finite difference scheme as U, but with $f(x_j, t_{m+1})$ replaced by T_j^{m+1}, $U_j^0 = u_0(x_j)$ replaced by $e_j^0 = 0$, and $u_1(x_j)$ replaced by T_j^1.

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Theorem 2 with U^m replaced by e^m, U^0 replaced by e^0 and $f(x_j, t_{k+1})$ replaced by T_j^{k+1} for $j = 1, \ldots, J-1$ and $k = 1, \ldots, M-1$, gives that

$$\mathcal{M}^2(e^m) \leq \mathrm{e}^2\,\mathcal{M}^2(e^0) + 2\,\mathrm{e}^2\,T\sum_{k=1}^{m}\Delta t\,\left\|T^{k+1}\right\|^2, \quad \text{for } m = 1, \ldots, M-1.$$

Hence, the global error e satisfies an identical finite difference scheme as U, but with $f(x_j, t_{m+1})$ replaced by T_j^{m+1}, $U_j^0 = u_0(x_j)$ replaced by $e_j^0 = 0$, and $u_1(x_j)$ replaced by T_j^1.

Theorem 2 with U^m replaced by e^m, U^0 replaced by e^0 and $f(x_j, t_{k+1})$ replaced by T_j^{k+1} for $j = 1, \ldots, J-1$ and $k = 1, \ldots, M-1$, gives that

$$\mathcal{M}^2(e^m) \leq \mathrm{e}^2\, \mathcal{M}^2(e^0) + 2\,\mathrm{e}^2\, T \sum_{k=1}^{m} \Delta t \left\| T^{k+1} \right\|^2, \quad \text{for } m = 1, \ldots, M-1.$$

It remains to bound the terms on the r.h.s. of this inequality.

Now, because $(J-1)\Delta x \leq (b-a)$, it follows from (7) that

$$\begin{aligned} \max_{1\leq k\leq m} \left\| T^{k+1} \right\|^2 &= \max_{1\leq k\leq m} \sum_{j=1}^{J-1} \Delta x \, |T_j^{k+1}|^2 \\ &\leq (b-a) \left[\frac{1}{12} c^2 \Delta x^2 M_{4x} + \frac{5}{3} \Delta t M_{3t} \right]^2 . \end{aligned}$$

Now, because $(J-1)\Delta x \leq (b-a)$, it follows from (7) that

$$\max_{1\leq k\leq m} \left\| T^{k+1} \right\|^2 = \max_{1\leq k\leq m} \sum_{j=1}^{J-1} \Delta x \, | T_j^{k+1} |^2$$
$$\leq (b-a) \left[\frac{1}{12} c^2 \Delta x^2 M_{4x} + \frac{5}{3} \Delta t M_{3t} \right]^2 .$$

Similarly,

$$\mathcal{M}^2(e^0) \leq (b-a) \left[\frac{1}{2} \Delta t \, M_{2t} \right]^2 + c^2 (b-a) \left[\frac{1}{2} \, \Delta t^2 M_{1x2t} \right]^2 .$$

Hence, finally,

$$\mathcal{M}^2(e^m) \leq \mathrm{e}^2(b-a)\left[\frac{1}{2}\Delta t\, M_{2t}\right]^2 + c^2\mathrm{e}^2(b-a)\left[\frac{1}{2}\,\Delta t^2 M_{1x2t}\right]^2 + 2\mathrm{e}^2\, T^2(b-a)\left[\frac{1}{12}c^2\Delta x^2 M_{4x} + \frac{5}{3}\Delta t M_{3t}\right]^2$$

for $m = 1, \ldots, M-1$.

Hence, finally,

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for $m = 1, \ldots, M-1$.

Thus, provided that M_{2t}, M_{1x2t}, M_{4x} and M_{3t} are all finite, we have that

$$\max_{m\in\{1,\ldots,M-1\}} [\mathcal{M}^2(u^m - U^m)]^{\frac{1}{2}} = \mathcal{O}(\Delta x^2 + \Delta t).$$

Hence, finally,

$$\mathcal{M}^2(e^m) \le \mathrm{e}^2(b-a)\left[\frac{1}{2}\Delta t\, M_{2t}\right]^2 + c^2\mathrm{e}^2(b-a)\left[\frac{1}{2}\,\Delta t^2 M_{1x2t}\right]^2 + 2\mathrm{e}^2\, T^2(b-a)\left[\frac{1}{12}c^2\Delta x^2 M_{4x} + \frac{5}{3}\Delta t M_{3t}\right]^2$$

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Thus, provided that M_{2t}, M_{1x2t}, M_{4x} and M_{3t} are all finite, we have that

$$\max_{m\in\{1,\ldots,M-1\}} [\mathcal{M}^2(u^m - U^m)]^{\frac{1}{2}} = \mathcal{O}(\Delta x^2 + \Delta t).$$

Note: The convergence of the scheme is *unconditional*; i.e., there is no limitation on the size of the time step Δt in terms of the spatial mesh-size Δx for the convergence of the sequence of numerical approximations to the solution of the wave equation to occur as $\Delta x, \Delta t \to 0$.

In Lecture 14 we shall study the explicit finite difference scheme for the wave equation. We will show that, in contrast with the implicit scheme, the explicit scheme is only *conditionally stable*, and its convergence will therefore also shown to be conditional; specifically, we shall require that

$$\frac{c\,\Delta t}{\Delta x} \leq c_0 < 1,$$

where c_0 is a positive constant and $c > 0$ is the wave speed, appearing as the coefficient of $\frac{\partial^2 u}{\partial x^2}$ in the wave equation.

Note: On the other hand, in contrast with the implicit scheme, when $f \equiv 0$ the explicit scheme will be shown to satisfy a discrete energy equality.