Numerical Solution of Partial Differential Equations

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Lecture 14

The explicit scheme: stability

Consider the closed interval $[a, b]$ of the real line, with $a < b$, and let $T > 0$. We shall be concerned with the finite difference approximation of the initial-boundary-value problem

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= f(x,t) && \text{for } (x,t) \in (a,b) \times (0, T], \\
u(x,0) &= u_0(x) && \text{for } x \in [a,b], \\
\frac{\partial u}{\partial t}(x,0) &= u_1(x) && \text{for } x \in [a,b], \\
u(a,t) = 0 \quad \text{and} \quad u(b,t) &= 0 && \text{for } t \in [0,T].
\end{aligned}
\tag{1}
$$

Here, f is assumed to be a continuous real-valued function defined on $(a, b) \times [0, T]$, u_0 and u_1 are supposed to be continuous real-valued functions defined on $[a, b]$, and we shall assume compatibility of the initial data with the boundary conditions, in the sense that u_0 and u_1 are required to vanish at both $x = a$ and $x = b$. As before, $c > 0$ is the wave speed.

For $M \geq 2$, we define $\Delta t := T/M$, and for $J \geq 2$ the spatial step is taken to be $\Delta x := (b-a)/J$. We let $x_j := a + j\Delta x$ for $j = 0, 1, \ldots, J$ and $t_m := m\Delta t$ for $m = 0, 1, \ldots, M$.

On the space-time mesh $\{(x_j, t_m) : 0 \leq j \leq J,\ 0 \leq m \leq M\}$ we consider the finite difference scheme

$$
\begin{aligned}
\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2} &= f(x_j, t_m) && \text{for} \left\{ \begin{array}{l} j = 1, \ldots, J-1, \\ m = 1, \ldots, M-1, \end{array} \right. \\
U_j^0 &= u_0(x_j) && \text{for } j = 0, 1, \ldots, J, \\
U_j^1 &= U_j^0 + \Delta t\, u_1(x_j) && \text{for } j = 1, 2, \ldots, J-1, \\
U_0^m = 0 \text{ and } \quad U_J^m &= 0 && \text{for } m = 1, \ldots, M.
\end{aligned}
\tag{2}
$$

Once the values of U_j^{m-1} and U_j^m, for $j = 0, \ldots, J$, have been computed (or have been specified by the initial data, in the case of $m = 1$), the subsequent values U_j^{m+1}, $j = 0, \ldots, J$, for $m = 1, \ldots, M - 1$, can be computed explicitly from (2), without having to solve systems of linear algebraic equations; hence the terminology *explicit scheme*.

Stability of the explicit scheme

It will transpire from the analysis that will follow that the explicit scheme is, unlike the implicit scheme, which was shown to be unconditionally stable, now only conditionally stable: we shall prove its stability in a certain 'energy norm', whose precise definition will emerge during the course of our analysis, — the stability condition for the explicit scheme being that $c\Delta t/\Delta x \leq c_0$, for some positive constant $c_0 \in (0, 1)$.

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Note: The good news is that now, when $f \equiv 0$, the explicit scheme (unlike the implicit scheme) exhibits conservation of this discrete energy.

The left-hand side of equality $(2)_1$ can be rewritten as

$$\begin{aligned}
&\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} - c^2 D_x^+ D_x^- U_j^m \\
&= \frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} + \frac{c^2 \Delta t^2}{4} D_x^+ D_x^- \frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} \\
&\qquad - c^2 D_x^+ D_x^- \frac{U_j^{m+1} + 2U_j^m + U_j^{m-1}}{4}
\end{aligned}$$

for $j = 1, \ldots, J-1$.

Insertion of this into $(2)_1$ then yields

$$\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{\Delta t^2} = c^2 D_x^+ D_x^- \frac{U_j^{m+1} + 2U_j^m + U_j^{m-1}}{4} + f(x_j, t_m) \tag{3}$$

for $j = 1, \ldots, J-1$, $m = 1, \ldots, M-1$, where I signifies the identity operator, which maps any mesh function defined on the spatial mesh $\{x_j : j = 1, \ldots, J-1\}$ into itself.

We shall consider the inner products

$$(U, V) := \sum_{j=1}^{J-1} \Delta x\, U_j\, V_j,$$

$$(U, V] := \sum_{j=1}^{J} \Delta x\, U_j\, V_j,$$

and the associated norms, respectively, $\| \cdot \|$ and $\| \cdot]|$, defined by

$$\|U\| := (U, U)^{\frac{1}{2}} \quad \text{and} \quad \|U]| := (U, U]^{\frac{1}{2}}.$$

We begin by noting that, for any $j \in \{0, \ldots, J\}$ and $m \in \{1, \ldots, M-1\}$:

$$
\begin{aligned}
U_j^{m+1} - U_j^{m-1} &= (U_j^{m+1} - U_j^m) + (U_j^m - U_j^{m-1}) \\
&= (U_j^{m+1} + U_j^m) - (U_j^m + U_j^{m-1}), \\
U_j^{m+1} - 2U_j^m + U_j^{m-1} &= (U_j^{m+1} - U_j^m) - (U_j^m - U_j^{m-1}), \\
U_j^{m+1} + 2U_j^m + U_j^{m-1} &= (U_j^{m+1} + U_j^m) + (U_j^m + U_j^{m-1}).
\end{aligned}
\tag{4}
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U_j^{m+1} - U_j^{m-1} &= (U_j^{m+1} - U_j^m) + (U_j^m - U_j^{m-1}) \\
&= (U_j^{m+1} + U_j^m) - (U_j^m + U_j^{m-1}), \\
U_j^{m+1} - 2U_j^m + U_j^{m-1} &= (U_j^{m+1} - U_j^m) - (U_j^m - U_j^{m-1}), \\
U_j^{m+1} + 2U_j^m + U_j^{m-1} &= (U_j^{m+1} + U_j^m) + (U_j^m + U_j^{m-1}).
\end{aligned} \tag{4}$$

We then take the $(\cdot, \cdot)$ inner product of (3) with $U^{m+1} - U^{m-1}$, making use of $(4)_3$ and $(4)_1$ on the left-hand side, and $(4)_4$ and $(4)_2$ on the right-hand side, together with the equalities

$$(\mathcal{D}(A - B), A + B) = (\mathcal{D}A, A) - (\mathcal{D}B, B),$$

$$(\mathcal{D}(A + B), A - B) = (\mathcal{D}A, A) - (\mathcal{D}B, B),$$

provided that the finite difference operator $\mathcal{D}$ satisfies $(\mathcal{D}A, B) = (\mathcal{D}B, A)$.

Thus we obtain the following equality:

$$
\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad - \left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \\
&= -c^2\left(-D_x^+ D_x^- \frac{U^{m+1} + U^m}{2}, \frac{U^{m+1} + U^m}{2}\right) \\
&\quad + c^2\left(-D_x^+ D_x^- \frac{U^m + U^{m-1}}{2}, \frac{U^m + U^{m-1}}{2}\right) \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}
$$

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&= -c^2\left(-D_x^+ D_x^- \frac{U^{m+1} + U^m}{2}, \frac{U^{m+1} + U^m}{2}\right) \\
&\quad + c^2\left(-D_x^+ D_x^- \frac{U^m + U^{m-1}}{2}, \frac{U^m + U^{m-1}}{2}\right) \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}$$

Next, we shall perform summations by parts in the first two terms on the right-hand side, using that, for any mesh-function V defined on $\{x_j : j = 0, \dots, J\}$ and such that $V_0 = V_J = 0$, one has

$$(-D_x^+ D_x^- V, V) = (D_x^- V, D_x^- V] = \| D_x^- V]|^2.$$

Using these equalities with $V = \frac{1}{2}(U^{m+1} + U^m)$ and $V = \frac{1}{2}(U^m + U^{m-1})$, we deduce that

$$
\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad - \left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \\
&= -c^2 \left(D_x^- \frac{U^{m+1} + U^m}{2}, D_x^- \frac{U^{m+1} + U^m}{2}\right] \\
&\quad + c^2 \left(D_x^- \frac{U^m + U^{m-1}}{2}, D^x_- \frac{U^m + U^{m-1}}{2}\right] \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}) \\
&= -c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2}\right]\right\|^2 + c^2 \left\| D_x^- \frac{U^m + U^{m-1}}{2}\right]\right\|^2 \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}
$$

This implies, following a minor rearrangement of terms, that

$$
\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right]\Big|^2 \\
&= \left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \qquad (5) \\
&\quad + c^2 \left\| D_x^- \frac{U^m + U^{m-1}}{2} \right]\Big|^2 \\
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&\quad + c^2 \left\| D_x^- \frac{U^{m+1}+U^m}{2} \right]\!\Big|^2 \\
&= \left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^m-U^{m-1}}{\Delta t}, \frac{U^m-U^{m-1}}{\Delta t}\right) \qquad (5) \\
&\quad + c^2 \left\| D_x^- \frac{U^m+U^{m-1}}{2} \right]\!\Big|^2 \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}
$$

The second term on the left-hand side of (5) is nonnegative, as is the second term on the right-hand side.

We would therefore like to ensure that first term on the left-hand side of (5) and the first term on the right-hand side are also nonnegative.

To do so, we shall make a small diversion to investigate this. Letting

$$V_j^m := \frac{U_j^{m+1} - U_j^m}{\Delta t}, \qquad j = 0, \ldots, J,$$

and noting that $V_0^m = V_J^m = 0$, it follows that

$$\begin{aligned}\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)V^m, V^m\right) &= \|V^m\|^2 + \frac{1}{4}c^2\Delta t^2 (D_x^+ D_x^- V^m, V^m) \\ &= \|V^m\|^2 - \frac{1}{4}c^2\Delta t^2 (D_x^- V^m, D_x^- V^m] \\ &= \|V^m\|^2 - \frac{1}{4}c^2\Delta t^2 \|D_x^- V^m]|^2.\end{aligned}$$

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$$\begin{aligned}\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)V^m, V^m\right) &= \|V^m\|^2 + \frac{1}{4}c^2\Delta t^2 (D_x^+ D_x^- V^m, V^m) \\ &= \|V^m\|^2 - \frac{1}{4}c^2\Delta t^2 (D_x^- V^m, D_x^- V^m] \\ &= \|V^m\|^2 - \frac{1}{4}c^2\Delta t^2 \|D_x^- V^m]|^2.\end{aligned}$$

The left-most expression in this chain of equalities will be nonnegative if and only of

$$\|V^m\|^2 - \frac{1}{4}c^2\Delta t^2 \|D_x^- V^m]|^2 \geq 0.$$

Our objective is to show that this can be guaranteed by requiring that $c\Delta t/\Delta x \leq c_0$ for some constant $c_0 \in (0, 1)$.

Noting that for any nonnegative real numbers α and β one has $(\alpha - \beta)^2 \le 2\alpha^2 + 2\beta^2$, it follows that

$$\begin{aligned} \|D_x^- V^m]\|^2 &= \sum_{j=1}^{J} \Delta x \, |D_x^- V_j^m|^2 = (\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m - V_{j-1}^m)^2 \\ &\le 2\,(\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m)^2 + (V_{j-1}^m)^2 = 4\,(\Delta x)^{-1} \sum_{j=1}^{J-1} (V_j^m)^2 \\ &= 4\,(\Delta x)^{-2} \sum_{j=1}^{J-1} \Delta x \, (V_j^m)^2 = \left(\frac{2}{\Delta x}\right)^2 \|V\|^2. \end{aligned}$$

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$$
\begin{aligned}
\|D_x^- V^m]\|^2 &= \sum_{j=1}^{J} \Delta x \, |D_x^- V_j^m|^2 = (\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m - V_{j-1}^m)^2 \\
&\leq 2\,(\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m)^2 + (V_{j-1}^m)^2 = 4\,(\Delta x)^{-1} \sum_{j=1}^{J-1} (V_j^m)^2 \\
&= 4\,(\Delta x)^{-2} \sum_{j=1}^{J-1} \Delta x \,(V_j^m)^2 = \left(\frac{2}{\Delta x}\right)^2 \|V\|^2.
\end{aligned}
$$

Thus we deduce that

$$
\left(\left(I + \frac{1}{4} c^2 \Delta t^2 D_x^+ D_x^-\right) V^m, V^m\right) \geq \left(1 - \left(\frac{c\,\Delta t}{\Delta x}\right)^2\right) \|V^m\|^2. \qquad (6)
$$

We shall therefore suppose that the following condition holds, referred to as a Courant–Friedrichs–Lewy (or CFL) condition: there exists a positive constant c_0 such that

$$\frac{c\,\Delta t}{\Delta x} \leq c_0 < 1. \tag{7}$$

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Assuming that (7) holds, we then have from (6) that

$$\left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^{m+1}-U^m}{\Delta t}, \frac{U^{m+1}-U^m}{\Delta t}\right)$$
$$\geq (1-c_0^2)\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|^2.$$

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We shall therefore proceed by assuming that (7) holds, and define the *nonnegative* expression

$$\mathcal{N}^2(U^m) := \left(\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\ + c^2\left\| D_x^- \frac{U^{m+1} - U^m}{2}\right]\Bigg\|^2.$$

With this notation (5) becomes

$$\mathcal{N}^2(U^m) = \mathcal{N}^2(U^{m-1}) + (f(\cdot, t_m), U^{m+1} - U^{m-1}). \tag{8}$$

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In the special case when f is identically zero (8) guarantees the stability of the explicit scheme under the CFL condition (7); indeed, (8) implies that

$$\mathcal{N}^2(U^m) = \mathcal{N}^2(U^0), \qquad \text{for all } m = 1, \ldots, M-1.$$

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One can check (please check this!) that the mapping

$$U \mapsto \max_{m \in \{0,\ldots,M-1\}} \left[\mathcal{N}^2(U^m)\right]^{1/2}$$

is a norm on the linear space of all mesh functions U defined on the space-time mesh $\{(x_j, t_m) : j = 0, 1, \ldots, J, \;\; m = 0, 1, \ldots, M\}$ such that $U_0^m = U_J^m = 0$ for all $m = 0, 1, \ldots, M$.

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Thus, if the CFL condition (7) holds and f is identically zero, the explicit scheme (2) is (conditionally) stable in this norm.