Numerical Solution of Partial Differential Equations

Endre Süli

Mathematical Institute
University of Oxford
2021

Lecture 16

Finite difference approximation of scalar nonlinear hyperbolic conservation laws

Nonlinear hyperbolic conservation laws and systems of nonlinear hyperbolic conservation laws arise in numerous areas of application, fluid dynamics being one such field.

Here, we shall confine ourselves to the simplest possible case of an initial-value problem for the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \tag{1}$$

where $u = u(x, t)$, subject to the initial condition $u(x, 0) = u_0(x)$, where $u_0 \in C^1(\mathbb{R})$ and has compact support, i.e. u_0 is identically zero outside a bounded closed interval of $\mathbb{R}$.

The real-valued function f will be assumed to be twice continuously differentiable on $\mathbb{R}$ and we shall suppose that $f(0) = f'(0) = 0$, and $f''(s) \geq 0$ for all $s \in \mathbb{R}$.

Under these hypotheses f' is a nondecreasing function, whereby $f'(s) \geq 0$ for all $s \geq 0$. We shall assume further that $|f'(s)| \leq f'(|s|)$ for all $s \in \mathbb{R}$.

For example $f(s) = \frac{1}{2}s^2$ and $f(s) = \frac{1}{4}s^4 + \frac{1}{2}s^2$ satisfy these hypotheses.

Assuming that there is a $T > 0$ such that a solution $u \in C^1(\mathbb{R} \times [0, T])$ to the initial-value problem exists, then thanks to the chain rule the equation (1) can be rewritten as

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0 \quad \text{for } (x, t) \in \mathbb{R} \times (0, T]. \tag{2}$$

Motivated by the construction of the first-order upwind scheme in the previous section, we decompose $f'(u)$ into its nonnegative and nonpositive parts, as follows:

$$f'(u) = [f'(u)]_+ + [f'(u)]_-,$$

where we have used the notation:

$$[x]_+ := \frac{1}{2}(x + |x|) \quad \text{and} \quad [x]_- := \frac{1}{2}(x - |x|).$$

Clearly,

$$x = [x]_+ + [x]_-, \quad |x| = [x]_+ - [x]_-, \quad [x]_+ \geq 0 \quad \text{and} \quad [x]_- \leq 0 \quad \forall\, x \in \mathbb{R}.$$

With this notation, we can rewrite (2) as follows:

$$\frac{\partial u}{\partial t} + [f'(u)]_+ \frac{\partial u}{\partial x} + [f'(u)]_- \frac{\partial u}{\partial x} = 0 \quad \text{for } (x, t) \in \mathbb{R} \times (0, T]. \tag{3}$$

We approximate (3) by the following finite difference scheme

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + [f'(U_j^m)]_+ \, D_x^- U_j^m + [f'(U_j^m)]_- \, D_x^+ U_j^m = 0, \; \begin{cases} j \in \mathbb{Z}, \\ m = 0, \dots, M-1, \end{cases}$$
$$U_j^0 = u_0(x_j), \quad j \in \mathbb{Z}, \tag{4}$$

where $\Delta t = T/M$, $M \geq 1$, and $\mathbb{Z}$ is the set of all integers.

We will show that, under a certain CFL condition which we shall state below, the sequence of finite difference approximations $\{U_j^m\}_{j \in \mathbb{Z},\, 0 \le m \le M}$ is bounded, in the sense that

$$\max_{1 \le k \le M} \|U^k\|_\infty \le \|U^0\|_\infty, \tag{5}$$

where now $\|V\|_\infty := \max_{j \in \mathbb{Z}} |V_j|$.

To this end, we rewrite $(4)_1$ as follows:

$$\begin{aligned} U_j^{m+1} &= U_j^m - \frac{[f'(U_j^m)]_+ \, \Delta t}{\Delta x}(U_j^m - U_{j-1}^m) - \frac{[f'(U_j^m)]_- \, \Delta t}{\Delta x}(U_{j+1}^m - U_j^m) \\ &= \left(1 - \frac{\Delta t}{\Delta x}\left([f'(U_j^m)]_+ - [f'(U_j^m)]_-\right)\right) U_j^m \\ &\qquad\qquad + \frac{[f'(U_j^m)]_+ \, \Delta t}{\Delta x} U_{j-1}^m + \frac{-[f'(U_j^m)]_- \, \Delta t}{\Delta x} U_{j+1}^m \\ &= \left(1 - \frac{|f'(U_j^m)| \, \Delta t}{\Delta x}\right) U_j^m + \frac{[f'(U_j^m)]_+ \, \Delta t}{\Delta x} U_{j-1}^m + \frac{-[f'(U_j^m)]_- \, \Delta t}{\Delta x} U_{j+1}^m \end{aligned} \tag{6}$$

for all $j \in \mathbb{Z}$ and all $m = 0, \ldots, M-1$. Suppose that the following CFL condition holds:

$$\frac{f'(\|U^0\|_\infty) \, \Delta t}{\Delta x} \leq 1. \tag{7}$$

Suppose further, as an inductive hypothesis, that, for some $m \geq 0$,

$$\frac{f'(\|U^k\|_\infty)\,\Delta t}{\Delta x} \leq 1 \qquad \text{for all } k = 0, \ldots, m. \tag{8}$$

Thanks to (7) this inductive hypothesis is satisfied for $m = 0$.

Suppose, for the inductive step, that (8) has already been shown to hold for some $m \geq 0$.

Because of the assumptions imposed on the function f, we have that $|f'(U_j^m)| \leq f'(|U_j^m|) \leq f'(\|U^m\|_\infty)$ for all $j \in \mathbb{Z}$.

It then follows from (8) with $k = m$ that

$$\frac{|f'(U_j^m)|\,\Delta t}{\Delta x} \leq 1 \qquad \text{for all } j \in \mathbb{Z},$$

and then (6) implies that

$$\begin{aligned} |U_j^{m+1}| &\leq \left(1 - \frac{|f'(U_j^m)|\,\Delta t}{\Delta x}\right)|U_j^m| + \frac{[f'(U_j^m)]_+\,\Delta t}{\Delta x}|U_{j-1}^m| + \frac{-[f'(U_j^m)]_-\Delta t}{\Delta x}|U_{j+1}^m| \\ &\leq \left(1 - \frac{|f'(U_j^m)|\,\Delta t}{\Delta x}\right)\|U^m\|_\infty + \frac{[f'(U_j^m)]_+\,\Delta t}{\Delta x}\|U^m\|_\infty + \frac{-[f'(U_j^m)]_-\,\Delta t}{\Delta x}\|U^m\|_\infty \\ &= \left(1 - \frac{|f'(U_j^m)|\,\Delta t}{\Delta x}\right)\|U^m\|_\infty + \frac{|f'(U_j^m)|\,\Delta t}{\Delta x}\|U^m\|_\infty = \|U^m\|_\infty \end{aligned}$$

for all $j \in \mathbb{Z}$. Therefore,

$$\|U^{m+1}\|_\infty \leq \|U^m\|_\infty. \tag{9}$$

To complete the inductive step it remains to show that (8) holds with m replaced by $m+1$.

By (9) and since f' is nondecreasing, we have for all $k = 0, \ldots, m$:

$$\frac{f'(\|U^{m+1}\|_\infty)\,\Delta t}{\Delta x} \leq \frac{f'(\|U^m\|_\infty)\,\Delta t}{\Delta x} \leq 1. \tag{10}$$

The inequality (10) shows that (8) holds with m replaced by $m+1$, which then completes the inductive step.

Thus we have shown that, under the CFL condition (7),

$$\|U^{m+1}\|_\infty \leq \|U^m\|_\infty \leq \cdots \leq \|U^0\|_\infty \qquad \text{for all } m = 0, 1, \ldots, M-1, \tag{11}$$

which completes the proof of the assertion that the sequence $\{U_j^m\}_{j\in\mathbb{Z},\, 0\leq m\leq M}$ of finite difference approximations generated by the scheme is bounded; in particular (5) has been shown to hold.

Assuming that u has continuous and bounded second partial derivatives with respect to x and t defined on $\mathbb{R} \times [0, T]$, it can be shown that

$$\max_{1 \le m \le M} \|u^m - U^m\|_\infty = \mathcal{O}(\Delta x + \Delta t),$$

as in the case of the linear first-order hyperbolic equation considered in the previous section, but we shall not include the proof of this result here.

One of the main difficulties in proving such an error bound is that now, unlike the linear first-order hyperbolic equation where a bound such as (11) would, thanks to the linearity of the finite difference scheme, automatically imply the stability of the scheme, in the case of the nonlinear partial differential equation considered here this is not the case: if $\{U_j^m\}$ and $\{V_j^m\}$ are two sequences of numerical solutions generated by the scheme from initial data $\{U_j^0\}$ and $\{V_j^0\}$ the inequality (11) does not automatically imply that

$$\|U^{m+1} - V^{m+1}\|_\infty \leq \|U^m - V^m\|_\infty \leq \cdots \leq \|U^0 - V^0\|_\infty$$

for all $m = 0, 1, \ldots, M-1$, which then complicates the convergence analysis of the finite difference scheme.