C4.3 Functional Analytic Methods for PDEs
Lecture 5

Luc Nguyen
luc.nguyen@maths

University of Oxford

MT 2021

In the last 4 lectures

- L^p spaces and their properties.

This lecture

- Pre-compactness criterion in $L^p(\Omega)$.
- Divergence theorem and Integration by parts formula.
- Weak derivatives.
- Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ as Banach spaces.
- Differentiation rule for convolution of Sobolev functions.

Pre-compactness criterion in $C(K)$

Theorem (Ascoli-Arzelà's theorem)

Let K be a compact subset of $\mathbb{R}^n$. Suppose that (f_i) is a sequence of functions of $C(K)$ such that

1. *(Boundedness)* $\sup_i \|f_i\|_{C(K)} < \infty$,
2. *(Equi-continuity) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for all i and all $x, y \in K$ with $|x - y| < \delta$.*

Then there exists a subsequence (f_{i_j}) which converges uniformly on K.

Pre-compactness criterion in $L^p(\Omega)$

Theorem (Kolmogorov-Riesz-Fréchet's theorem)

Let $1 \leq p < \infty$ and Ω be an open bounded subset of $\mathbb{R}^n$. Suppose that a sequence (f_i) of $L^p(\Omega)$ satisfies

1. *(Boundedness)* $\sup_i \|f_i\|_{L^p(\Omega)} < \infty$,
2. *(Equi-continuity in L^p) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p(\Omega)} < \varepsilon$ for all $|y| < \delta$, where $\tilde{f}_i$ is the extension by zero of f_i to the whole of $\mathbb{R}^n$.*

Then, there exists a subsequence (f_{i_j}) which converges in $L^p(\Omega)$.

By definition $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{f}_i = f_i$ in Ω and $\tilde{f}_i = 0$ in $\mathbb{R}^n \setminus \Omega$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- As in the proof of Ascoli-Arzelà's theorem, it suffices to show that, for every given $\varepsilon > 0$, there exists a subsequence $(f_{i_j}^{\varepsilon})$ of (f_i) such that

$$\|f_{i_j}^{\varepsilon} - f_{i_k}^{\varepsilon}\|_{L^p(\Omega)} \leq 3\varepsilon \text{ for large } j, k. \qquad (***)$$

- Claim: For every fixed $\varphi \in C_c^{\infty}(\mathbb{R}^n)$, the sequence $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
  - ⋆ First, by Hölder's inequality, we have

$$\|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} = \|f_i\|_{L^1(\Omega)} \leq \|f_i\|_{L^p(\Omega)} |\Omega|^{1/p'}.$$

    Thus, by the boundedness of (f_i) in $L^p(\Omega)$, we have that $(\tilde{f}_i)$ is bounded in $L^1(\mathbb{R}^n)$.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- Claim: $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
  - ⋆ $\sup_i \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} < \infty$.
  - ⋆ By Young's convolution inequality

$$\|\tilde{f}_i * \varphi\|_{L^\infty(\mathbb{R}^n)} \leq \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^\infty(\mathbb{R}^n)}.$$

    So $\sup_i \|\tilde{f}_i * \varphi\|_{C(\bar{\Omega})} < \infty$.
  - ⋆ Next,

$$\begin{aligned} |\tilde{f}_i * \varphi(x) - \tilde{f}_i * \varphi(y)| &\leq \int_{\mathbb{R}^n} \Big|\varphi(x-z) - \varphi(y-z)\Big| |\tilde{f}(z)|\, dz \\ &\leq \|\varphi\|_{Lip(\mathbb{R}^n)} |x-y| \|\tilde{f}_i\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

    So by squeezing $|x-y|$, we can make $\sup_i |\tilde{f}_i * \varphi(x) - \tilde{f}_i * \varphi(y)|$ as small as we want.

Pre-compactness criterion in $L^p(\Omega)$

Proof

- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- Now, take a non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and, for $\eta > 0$, let $\varrho_\eta(x) = \frac{1}{\eta^n}\varrho(x/\eta)$ be the standard mollifiers. Recall that we have the estimate

$$\begin{aligned}
\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p}^p &\leq \int_{\mathbb{R}^n} |\varrho_\eta(y)| \|\tau_{-y}\tilde{f}_i - \tilde{f}_i\|_{L^p}^p \, dy \\
&\leq \sup_{|y|\leq\eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}^p \int_{\mathbb{R}^n} |\varrho_\eta(y)| \, dy \\
&= \sup_{|y|\leq\eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}^p.
\end{aligned}$$

Pre-compactness criterion in $L^p(\Omega)$

Proof

- $(\tilde{f}_i * \varphi|_{\bar{\Omega}})$ satisfies the condition of Ascoli-Arzelà's theorem.
- $\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p} \leq \sup_{|y|\leq\eta} \|\tau_y \tilde{f}_i - \tilde{f}_i\|_{L^p}$.
- We are now ready to prove (***):
  - ⋆ By the equi-continuity, there exists a small $\eta > 0$ such that $\|\tilde{f}_i * \varrho_\eta - \tilde{f}_i\|_{L^p} \leq \varepsilon$ for all i.
  - ⋆ Using Ascoli-Arzelà's theorem, select a subsequence $(f_{i_j}^\varepsilon)$ of (f_i) such that $(\tilde{f}_{i_j}^\varepsilon * \varrho_\eta|_{\bar{\Omega}})$ is convergent in $C(\bar{\Omega})$.
  - ⋆ It follows that $\|\tilde{f}_{i_j}^\varepsilon * \varrho_\eta - \tilde{f}_{i_k}^\varepsilon * \varrho_\eta\|_{L^p(\Omega)} \leq \varepsilon$ for large j, k.
  - ⋆ Consequently, by triangle inequality,

$$\begin{aligned}\|f_{i_j}^\varepsilon - f_{i_k}^\varepsilon\|_{L^p(\Omega)} \leq \|f_{i_j}^\varepsilon * \varrho_\eta - f_{i_k}^\varepsilon * \varrho_\eta\|_{L^p(\Omega)} + \|f_{i_j}^\varepsilon * \varrho_\eta - f_{i_j}^\varepsilon\|_{L^p(\Omega)} \\ + \|f_{i_k}^\varepsilon * \varrho_\eta - f_{i_k}^\varepsilon\|_{L^p(\Omega)} \leq 3\varepsilon \text{ for large } j, k,\end{aligned}$$

  which is (***).

Frequently used terminologies/notations

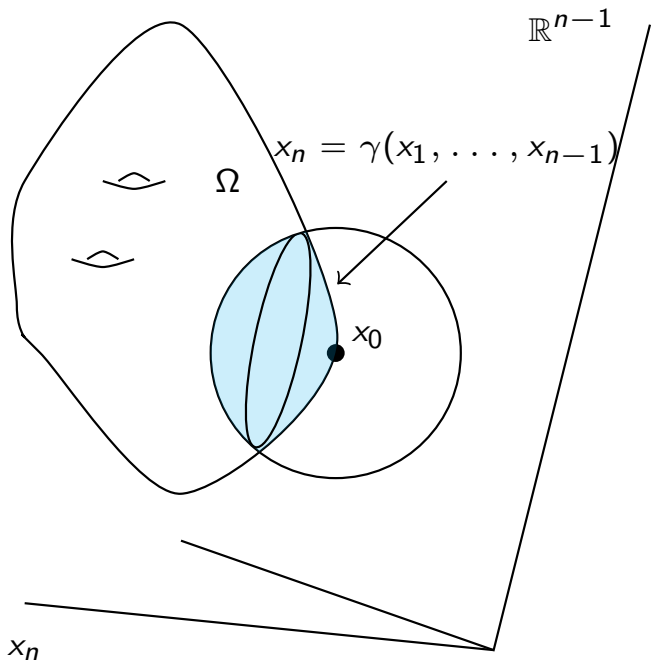
- Ω denotes a domain in $\mathbb{R}^n$.
- $C^k(\Omega)$ denotes the space of functions which are k-times continuously differentiable in Ω.
- $C^k(\bar{\Omega})$ denotes the subspace of $C^k(\Omega)$ consisting of functions which can be extended to a k-times continuously differentiable functions on some open set containing $\bar{\Omega}$.
- $C_c^k(\Omega)$ denotes the subspace of $C^k(\Omega)$ consisting of functions f such that $Supp(f) = \overline{\{f \neq 0\}}$ is a bounded closed subset of Ω.

Frequently used terminologies/notations

- Ω is said to be a Lipschitz (resp. C^k) domain, or equivalently, $\partial\Omega$ is said to be Lipschitz (resp. C^k), if for every $x_0 \in \partial\Omega$ there exists a radius $r_0 > 0$ such that, after a relabeling of coordinate axes if necessary,

$$\Omega \cap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

for some Lipschitz (resp. C^k) function γ.



Divergence theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Fact: $\partial\Omega$ admits an 'outward pointing' unit normal n.

Theorem (Divergence theorem)

Let $F \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Then

$$\int_\Omega div\, F\, dx = \int_{\partial\Omega} F \cdot n\, dS.$$

In particular, if $F \in C^1_c(\Omega; \mathbb{R}^n)$, then

$$\int_\Omega div\, F\, dx = 0.$$

IBP formula

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.

Theorem (Integration by parts formula)

Let $f, g \in C^1(\bar{\Omega})$. Then

$$\int_\Omega f\, \partial_i g\, dx = \int_{\partial\Omega} f g n_i\, dS - \int_\Omega \partial_i f\, g\, dx.$$

In particular, if f or g has compact support in Ω, then

$$\int_\Omega f\, \partial_i g\, dx = -\int_\Omega \partial_i f\, g\, dx.$$

Weak derivatives

Let Ω be a domain in $\mathbb{R}^n$.

Definition

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. A function $g \in L^1_{loc}(\Omega)$ is said to be a weak α-derivative of f if

$$\int_\Omega f \, \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{1}$$

We write $g = \partial^\alpha f$ in the weak sense.

The function φ is called a *test function*.

Example of weak derivatives

- If $f \in C^1(\bar{\Omega})$ and Ω is a bounded Lipschitz domain, then its classical derivatives are also its weak derivatives.
- Suppose $\Omega = (-1, 1)$ and $f(x) = |x|$. Then, if $\varphi \in C_c^\infty(-1, 1)$, we have by IBP that

$$\begin{aligned}\int_{-1}^{1} f(x)\,\varphi'(x)\,dx &= \int_{-1}^{0} (-x)\varphi'(x)\,dx + \int_{0}^{1} x\varphi'(x)\,dx \\ &= -x\varphi(x)\Big|_{-1}^{0} - \int_{-1}^{0} (-1)\varphi(x)\,dx \\ &\quad + x\varphi(x)\Big|_{0}^{1} - \int_{0}^{1} (1)\varphi(x)\,dx \\ &= -\int_{-1}^{1} \operatorname{sign}(x)\varphi(x)\,dx.\end{aligned}$$

So $f'(x) = \operatorname{sign}(x)$ in the weak sense.

Uniqueness of weak derivatives

Lemma

Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index. The weak α-derivative of f, if exists, is uniquely defined up to a set of measure zero.

This follows from the definition of weak derivative and the following:

Lemma (Fundamental lemma of the Calculus of Variations)

Let $g \in L^1_{loc}(\Omega)$. If $\int_\Omega g\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $g = 0$ a.e. in Ω.

Uniqueness of weak derivatives

Proof

- We will only consider the case Ω is a bounded domain and $g \in L^1(\Omega)$. The general case is left as an exercise.
- In Sheet 1, you showed that $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$. Thus, for any $\varepsilon > 0$, we can select $h \in C_c^\infty(\Omega)$ such that $\|g - h\|_{L^1} \leq \varepsilon$. Furthermore, by triangle inequality $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$.
- For $\delta > 0$, let $h_\delta = \frac{h}{\sqrt{\delta^2 + h^2}}$ so that $h_\delta \in C_c^\infty(\Omega)$ and $|h_\delta| \leq 1$.
- By hypotheses, $\int_\Omega g h_\delta \, dx = 0$.
- By construction, $\left| \int_\Omega (g - h) h_\delta \, dx \right| \leq \|g - h\|_{L^1} \|h_\delta\|_{L^\infty} \leq \varepsilon$.
- It follows that

$$\varepsilon \geq \int_\Omega g h_\delta \, dx - \int_\Omega (g - h) h_\delta \, dx = \int_\Omega h h_\delta \, dx.$$

Uniqueness of weak derivatives

Proof

- Recalling the expression of h_δ, we have

$$\varepsilon \geq \int_\Omega \frac{h^2}{\sqrt{\delta^2 + h^2}}\, dx.$$

- The integrand on the right hand side converges monotonically increasingly to $|h|$. Thus, by Lebesgue's monotone convergence theorem,

$$\varepsilon \geq \int_\Omega |h|\, dx = \|h\|_{L^1}.$$

- Recall that $\|h\|_{L^1} \geq \|g\|_{L^1} - \varepsilon$, we obtain that $2\varepsilon \geq \|g\|_{L^1}$. Sending $\varepsilon \to 0$, we obtain $\|g\|_{L^1} = 0$, i.e. $g = 0$ a.e. in Ω.

A relation between classical and weak derivatives

Remark

Suppose that

(i) $f \in L^1(\Omega)$ *is weakly differentiable with weak derivatives* $\partial_1^w f$, $\ldots$, $\partial_n^w f$,

(ii) *and, for some subdomain* $\omega \subset \Omega$, f *is classically differentiable in* ω *with classical derivatives* $\partial_1^c f$, $\ldots$, $\partial_n^c f$.

Then

$$\partial_i^w f = \partial_i^c f \text{ a.e. in } \omega \text{ for all } i = 1, \ldots, n.$$

A relation between classical and weak derivatives

Sketch of proof

- Using the definition of weak derivatives, $f|_\omega$ is weakly differentiable with weak derivatives $\partial_1^w f|_\omega, \ldots, \partial_n^w f|_\omega$.
- As f is classically differentiable in ω, its classical derivatives are also weak derivatives of $f|_\omega$.
- By the uniqueness of weak derivatives, the conclusion follows.

Example of non-existence of weak derivatives

If $\Omega = (-1, 1)$ and $u(x) = \operatorname{sign}(x)$, then u has no weak derivative.
Proof

- Suppose otherwise that $u' = g \in L^1_{loc}(-1, 1)$. Then, for $\varphi \in C_c^\infty(-1, 1)$,

$$\begin{aligned}\int_{-1}^{1} g(x)\varphi(x)\,dx &= \int_{-1}^{0} \varphi'(x)\,dx - \int_{0}^{1} \varphi'(x)\,dx \\ &= [\varphi(0) - \varphi(-1)] - [\varphi(1) - \varphi(0)] \\ &= 2\varphi(0).\end{aligned}$$

- In particular, if we take $\underline{\varphi \in C_c^\infty(-1, 0)}$, we have $\int_{-1}^{0} g(x)\varphi(x)\,dx = 0$. So $g = 0$ a.e. in $(-1, 0)$. Likewise, $g = 0$ a.e. in $(0, 1)$. So $g = 0$ a.e. in $(-1, 1)$.
- We thus have $0 = \int_{-1}^{1} g(x)\varphi(x)\,dx = 2\varphi(0)$ for all $\varphi \in C_c^\infty(-1, 1)$, which is impossible.

The Sobolev spaces $W^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p \leq \infty$, define

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \Big| \forall |\alpha| \leq k, \text{ the weak derivative } \partial^\alpha f \text{ exists and belongs to } L^p(\Omega) \Big\}.$$

We equip $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big[\sum_{|\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \Big]^{\frac{1}{p}}$$

so that $W^{k,p}(\Omega)$ is a normed vector space (check this!).

- For $p = 2$, we also write $H^k(\Omega)$ for $W^{k,2}(\Omega)$. These are inner product spaces (check this!) with inner product

$$\langle u, v \rangle_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

Examples of Sobolev functions

Let $\Omega = (-1, 1)$ and $f(x) = |x|$.

- We have that $f'(x) = \operatorname{sign}(x)$ and so $f \in W^{1,p}(-1, 1)$ for every $p \in [1, \infty]$.
- The function $f'(x) = \operatorname{sign}(x)$ has no weak derivatives, and so $f \notin W^{2,p}(-1, 1)$ for any $p \in [1, \infty]$.

Completeness of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is a Banach space. When $p = 2$, $W^{k,2}(\Omega)$ is a Hilbert space.

Proof

- We have seen that $W^{k,p}$ is a normed vector space and $W^{k,2}$ is an inner product space. It remains to show that $W^{k,p}$ is complete.
- Suppose that (u_m) is a Cauchy sequence in $W^{k,p}$. We need to show that there exists $u \in W^{k,p}$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ is Cauchy in L^p, as

$$\|\partial^\alpha u_m - \partial^\alpha u_j\|_{L^p} \leq \|u_m - u_j\|_{W^{k,p}}.$$

  By Riesz-Fischer's theorem, we have that $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.

Completeness of $W^{k,p}(\Omega)$

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- To conclude, we show that $u := v_{(0,\dots,0)}$ belongs to $W^{k,p}$ and $u_m \to u$ in $W^{k,p}$.
  - ⋆ By definition of weak derivatives, we have for $|\alpha| \leq k$ that

$$\int_\Omega u_m \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \, \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega),$$

  - ⋆ Now we would like to pass $m \to \infty$. By Hölder's inequality

$$\left| \int_\Omega (u_m - u) \partial^\alpha \varphi \, dx \right| \leq \|u_m - u\|_{L^p} \|\partial^\alpha \varphi\|_{L^{p'}} \to 0.$$

  So $\int_\Omega u_m \partial^\alpha \varphi \, dx \to \int_\Omega u \partial^\alpha \varphi \, dx$.
  - ⋆ Similarly, $\int_\Omega \partial^\alpha u_m \, \varphi \, dx \to \int_\Omega v_\alpha \varphi \, dx$.

Completeness of $W^{k,p}(\Omega)$

Proof

- (u_m) is Cauchy in $W^{k,p}$.
- For $|\alpha| \leq k$, $(\partial^\alpha u_m)$ converges in L^p to some $v_\alpha \in L^p$.
- ⋆ We thus have

$$\int_\Omega u \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  So v_α is the weak α-derivative of u. So $u \in W^{k,p}$.

- ⋆ Now

$$\begin{aligned} \|u_m - u\|_{W^{k,p}}^p &= \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - \partial^\alpha u\|_{L^p}^p \\ &= \sum_{|\alpha| \leq k} \|\partial^\alpha u_m - v_\alpha\|_{L^p}^p \overset{m \to \infty}{\longrightarrow} 0. \end{aligned}$$

  So $u_m \to u$ in $W^{k,p}$.

- We conclude that $W^{k,p}$ is complete.