C4.3 Functional Analytic Methods for PDEs
Lecture 6

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In the last lecture

- Divergence theorem and Integration by parts formula.
- Definition of weak derivatives and
- Sobolev spaces $W^{k,p}(\Omega)$

This lecture

- Dual of $W^{1,p}$
- Sobolev spaces $W_0^{k,p}(\Omega)$.
- Differentiation rule for convolution of Sobolev functions.
- Density results for Sobolev spaces.
- Extension theorems for Sobolev functions.

Reflexivity of $W^{k,p}(\Omega)$

Theorem

For $k \geq 0$ and $1 < p < \infty$, $W^{k,p}(\Omega)$ is reflexive.

Proof

- We will only consider the case $k = 1$. The general case requires some minor changes.
- By Eberlein's theorem, we only need to show that every bounded sequence in $W^{1,p}$ has a weakly convergent subsequence.
- Suppose $(u_m) \subset W^{1,p}$ is bounded. Then, (u_m) and $(\partial_i u_m)$ are bounded in L^p.
- By the weak sequential compactness property of L^p for $1 < p < \infty$, there exists a subsequence (u_{m_j}) such that (u_{m_j}) and $(\partial_i u_{m_j})$ are weakly convergent in L^p. Let u be the L^p weak limit of (u_{m_j}) and v_i be the L^p weak limit of $(\partial_i u_{m_j})$.

Reflexivity of $W^{k,p}(\Omega)$

- To conclude, we show that u belongs to $W^{1,p}$ and $u_{m_j} \rightharpoonup u$ in $W^{1,p}$.
- The proof that $u \in W^{1,p}$ is similar to the one we did moment ago, but also has some subtle difference: By definition of weak derivatives, we have

$$\int_\Omega u_{m_j} \partial_i \varphi = - \int_\Omega \partial_i u_{m_j} \, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega),$$

Sending $j \to \infty$ by using the definition weak convergence, we obtain

$$\int_\Omega u \partial_i \varphi = - \int_\Omega v_i \, \varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

So $v_i = \partial_i u$ in the weak sense. So $u \in W^{1,p}$.

Reflexivity of $W^{k,p}(\Omega)$

- It remains to show that, if $A \in (W^{1,p})^*$, then $Au_{m_j} \to Au$.
  - ⋆ Define $E : W^{1,p}(\Omega) \to (L^p(\Omega))^{n+1}$ by $Ef = (f, \partial_1 f, \ldots, \partial_n f)$. Then E is an isometry.
  - ⋆ Let $X := E(W^{1,p}(\Omega))$ and $Y := (L^p(\Omega))^{n+1}$. Define $\tilde{A} : X \to \mathbb{R}$ by $\tilde{A}p = AE^{-1}p$ for $p \in X$. Then $\tilde{A} \in X^*$. By Hahn-Banach's theorem, it has an extension $\hat{A} \in Y^*$.
  - ⋆ It follows that

$$\begin{aligned} Au_{m_j} &= \tilde{A}Eu_{m_j} = \hat{A}Eu_{m_j} \\ &= \hat{A}(u_{m_j}, 0, \ldots, 0) + \sum_i \hat{A}(0, 0, \ldots, 0, \partial_i u_{m_j}, 0, \ldots, 0) \\ &=: B(u_{m_j}) + \sum_i B_i(\partial_i u_{m_j}) \\ &\to B(u) + \sum_i B_i(\partial_i u) = Au. \end{aligned}$$

This concludes the proof.

The Sobolev spaces $W_0^{k,p}(\Omega)$

- Ω: a domain of $\mathbb{R}^n$.
- For $k \geq 0$ and $1 \leq p < \infty$, define

$$W_0^{k,p}(\Omega) = \text{the closure of } C_c^\infty(\Omega) \text{ in } W^{k,p}(\Omega).$$

When $p = 2$, we also write $H_0^k(\Omega)$ for $W_0^{k,2}(\Omega)$.

- In other words, $u \in W_0^{k,p}(\Omega)$ if there exist $u_m \in C_c^\infty(\Omega)$ such that $\|u_m - u\|_{W^{k,p}} \to 0$.
- When $k = 0$, $1 \leq p < \infty$, and Ω is a bounded domain, we have seen in Sheet 1 that $W_0^{0,p}(\Omega) = W^{0,p}(\Omega) = L^p(\Omega)$.
  In general, this is not true for $k \geq 1$. Roughly speaking, $W_0^{k,p}(\Omega)$ consists of functions f in $W^{k,p}(\Omega)$ such that

$$\text{`}\partial^\alpha f = 0 \text{ on } \partial\Omega\text{' for all } |\alpha| \leq k - 1.$$

IBP formula for Sobolev functions

Proposition (Integration by parts)

Let $u \in W^{k,p}(\Omega)$ and $v \in W_0^{k,p'}(\Omega)$ with $k \geq 0$, $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_\Omega \partial^\alpha u v \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v \, dx \text{ for all } |\alpha| \leq k.$$

Proof

- By definition of $W_0^{k,p'}$, there exists $v_m \in C_c^\infty(\Omega)$ such that $v_m \to v$ in $W^{k,p'}$. In particular, $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- By the definition of weak derivatives,

$$\int_\Omega \partial^\alpha u v_m \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m \, dx \text{ for all } |\alpha| \leq k.$$

IBP formula for Sobolev functions

Proof

- $\partial^\alpha v_m \to \partial^\alpha v$ in $L^{p'}$ for all $|\alpha| \leq k$.
- $\int_\Omega \partial^\alpha u v_m \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m \, dx$ for all $|\alpha| \leq k$.
- We can now pass $m \to \infty$ as in the proof of the completeness of Sobolev spaces.
  - ⋆ By Hölder's inequality

$$\left| \int_\Omega \partial^\alpha u (v_m - v) \, dx \right| \leq \|\partial^\alpha u\|_{L^p} \|v_m - v\|_{L^{p'}} \to 0.$$

    So $\int_\Omega \partial^\alpha u v_m \, dx \to \int_\Omega \partial^\alpha u v \, dx$.
  - ⋆ Similarly, $\int_\Omega u \partial^\alpha v_m \, dx \to \int_\Omega u \partial^\alpha v \, dx$.
  - ⋆ We conclude that

$$\int_\Omega \partial^\alpha u v \, dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v \, dx.$$

Differentiation rule for convolution of Sobolev functions

- Suppose $k \geq 0$ and $1 \leq p < \infty$.
- Let $f \in L^p(\mathbb{R}^n)$ and $g \in C_c^k(\mathbb{R}^n)$. We knew that $f * g \in C^k(\mathbb{R}^n)$ and

$$\partial^\alpha(f * g) = f * (\partial^\alpha g) \text{ for all } |\alpha| \leq k.$$

Lemma

Assume $f \in W^{k,p}(\mathbb{R}^n)$ and $g \in C_c^k(\mathbb{R}^n)$ for some $k \geq 0$ and $1 \leq p < \infty$, then

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g \textit{ for all } |\alpha| \leq k.$$

Differentiation rule for convolution of Sobolev functions

Proof

- We will only consider the case $k = 1$. We aim to prove that

$$\partial_{x_1}(f * g) = (\partial_{x_1} f) * g$$

- We compute

$$\begin{aligned}\partial_{x_1}(f * g)(x) = f * (\partial_{x_1} g)(x) &= \int_{\mathbb{R}^n} f(y)\,\partial_{x_1} g(x - y)\,dy \\ &= -\int_{\mathbb{R}^n} f(y)\,\partial_{y_1} g(x - y)\,dy \\ &= \int_{\mathbb{R}^n} \partial_{y_1} f(y)\,g(x - y)\,dy = ((\partial_{x_1} f) * g)(x).\end{aligned}$$

  So we are done.

Approximation of identity in Sobolev spaces

Theorem (Approximation of identity)

Let ϱ *be a non-negative function in* $C_c^\infty(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varrho = 1$. *For* $\varepsilon > 0$, *let*

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\Big(\frac{x}{\varepsilon}\Big).$$

If $f \in W^{k,p}(\mathbb{R}^n)$ *for some* $k \geq 0$ *and* $1 \leq p < \infty$, *then* $f * \varrho_\varepsilon \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ *and*

$$\lim_{\varepsilon \to 0} \|f * \varrho_\varepsilon - f\|_{W^{k,p}(\mathbb{R}^n)} = 0.$$

In particular $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$.

Approximation of identity in Sobolev spaces

Proof

- Let $f_\varepsilon = f * \varrho_\varepsilon$.
  - ⋆ As $\varrho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, we have $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.
  - ⋆ As $f \in L^p(\mathbb{R}^n)$ and $\varrho_\varepsilon \in L^1(\mathbb{R}^n)$, Young's inequality gives that $f_\varepsilon \in L^p(\mathbb{R}^n)$.
  - ⋆ The approximation of identity theorem in L^p gives that $\|f_\varepsilon - f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- By the differentiation rule for convolution of Sobolev functions, we have $\partial^\alpha f_\varepsilon = (\partial^\alpha f) * \varrho_\varepsilon$ for $|\alpha| \leq k$. Repeat the argument as above, we have $\partial^\alpha f_\varepsilon \in L^p(\mathbb{R}^n)$ and $\|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p} \to 0$ as $\varepsilon \to 0$.
- We deduce that $f_\varepsilon \in W^{k,p}(\mathbb{R}^n)$ and

$$\|f_\varepsilon - f\|_{W^{k,p}} = \Big[\sum_{|\alpha| \leq k} \|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{L^p}^p\Big]^{1/p} \xrightarrow{\varepsilon \to 0} 0.$$

Meyers-Serrin's theorem

Theorem (Meyers-Serrin)

Suppose Ω is a domain in $\mathbb{R}^n$, $k \geq 0$ and $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. Namely, for every $u \in W^{k,p}(\Omega)$ there exists a sequence $(u_m) \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that u_m converges to u in $W^{k,p}(\Omega)$.

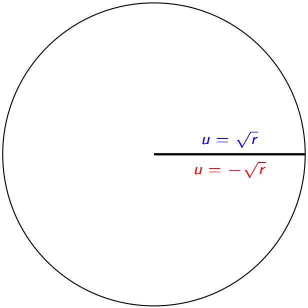
Remark: No regularity on Ω is assumed.

A question and an obstruction

Question

Is $C^\infty(\bar\Omega) \cap W^{k,p}(\Omega)$ dense in $W^{k,p}(\Omega)$?

Answer: Not always.

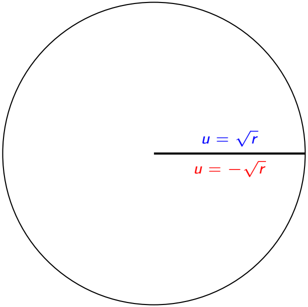


$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$
$\bar\Omega = \{x^2 + y^2 \leq 1\}$

Consider $u(x,y) = \sqrt{r}\cos\frac{\theta}{2}$ where $(x,y) = (r\cos\theta, r\sin\theta)$.
$u \in C^\infty(\Omega)$.
u is discontinuous in $\bar\Omega$.
One computes

$$\|u\|_{L^2}^2 = \int_\Omega u^2\,dx\,dy = \int_0^1 \int_0^{2\pi} r\cos^2\frac{\theta}{2}\, r\,dr\,d\theta = \frac{\pi}{3},$$

A question and an obstruction



$$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$$
$$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$$
$$D = \{x^2 + y^2 < 1\}$$

Consider $u(x,y) = \sqrt{r} \cos \frac{\theta}{2}$.
$u \in C^\infty(\Omega)$ and $u \notin C(\bar{\Omega})$.
One computes $\|u\|_{L^2}^2 = \frac{\pi}{3}$,

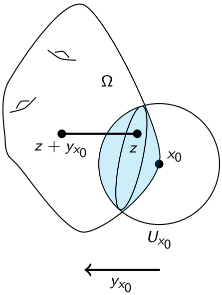
$$|\nabla u|^2 = (\partial_r u)^2 + \frac{1}{r^2}(\partial_\theta u)^2 = \frac{1}{4r},$$

$$\|\nabla u\|_{L^2}^2 = \int_\Omega |\nabla u|^2 \, dx \, dy$$
$$= \int_0^1 \int_0^{2\pi} \frac{1}{4r} \, r \, dr \, d\theta = \frac{\pi}{2},$$

So $u \in W^{1,2}(\Omega)$.
The jump discontinuity across $\theta = 0$ is an obstruction to approximate u by functions in $C^\infty(\bar{\Omega})$. It is in fact not possible, as $u \notin W^{1,2}(D)$.

The segment condition

- Ω: a domain in $\mathbb{R}^n$.
- Ω is said to satisfy the segment condition if every $x_0 \in \partial\Omega$ has a neighborhood U_{x_0} and a non-zero vector y_{x_0} such that if $z \in \bar{\Omega} \cap U_{x_0}$, then $z + ty_{x_0} \in \Omega$ for all $t \in (0, 1)$.



- Note that if Ω is Lipschitz, then it satisfies the segment condition.

Approximation by functions in $C^\infty(\bar\Omega)$

Theorem (Global approximation by functions smooth up to the boundary)

Suppose $k \geq 1$ and $1 \leq p < \infty$. If Ω satisfies the segment condition, then the set of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\Omega)$. In particular $C^\infty(\bar\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

- An important consequence of the theorem is the statement that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ when $1 \leq p < \infty$. In order words $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$.
- You will do the special when Ω is star-shaped in Sheet 2.

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Lemma

Assume that $k \geq 0$ and $1 \leq p < \infty$. If $u \in W_0^{k,p}(\Omega)$, then its extension by zero $\bar{u}$ to $\mathbb{R}^n$ belongs to $W_0^{k,p}(\mathbb{R}^n)$.

Proof

- Suppose $u \in W_0^{k,p}(\Omega)$ and let $\bar{u}$ be its extension by zero to $\mathbb{R}^n$. It is tempted to say that, as $\bar{u} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$,

$$\partial^\alpha \bar{u} = \begin{cases} \partial^\alpha u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \qquad (*)$$

  which belongs to $L^p(\mathbb{R}^n)$, and call it the end of the proof. For this to work, we need to show first that $\bar{u}$ is weakly differentiable!

Extension by zero of functions in $W_0^{k,p}(\Omega)$

Proof

- Let $(u_m) \subset C_c^\infty(\Omega)$ be such that $u_m \to u$ in $W^{k,p}(\Omega)$. Let $\bar{u}_m$ be the extension by zero of u_m to $\mathbb{R}^n$. Then $\bar{u}_m \in C_c^\infty(\mathbb{R}^n)$ and

$$\|\bar{u}_m - \bar{u}_j\|_{W^{k,p}(\mathbb{R}^n)} = \|u_m - u_j\|_{W^{k,p}(\Omega)} \xrightarrow{m,j\to\infty} 0.$$

- We thus have that $(\bar{u}_m)$ is Cauchy in $W^{k,p}(\mathbb{R}^n)$ and thus converges in $W^{k,p}$ to some $\bar{u}_* \in W^{k,p}(\mathbb{R}^n)$.
- To conclude, we show that $\bar{u}_* = \bar{u}$ a.e. in $\mathbb{R}^n$.
  - ⋆ As $\bar{u}_m$ converges to $\bar{u}_*$ in $L^p(\mathbb{R}^n)$, there is a subsequence $\bar{u}_{m_j}$ which converges a.e. to $\bar{u}_*$ in $\mathbb{R}^n$. This implies that $\bar{u}_* = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and u_{m_j} converges a.e. to $\bar{u}_*$ in Ω.
  - ⋆ Likewise, as u_{m_j} converges to u in $L^p(\Omega)$, we can extract yet another subsequence $u_{m_{j_l}}$ which converges a.e. to u in Ω. It follows that $\bar{u}_* = u$ a.e. in Ω.
  - ⋆ So $\bar{u} = \bar{u}_*$ a.e. in $\mathbb{R}^n$.

More on extension

Theorem (Stein's extension theorem)

Assume that Ω is a bounded Lipschitz domain. Then there exists a linear operator E sending functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ such that for every $k \geq 0$, $1 \leq p < \infty$ and $u \in W^{k,p}(\Omega)$ it hold that $Eu = u$ a.e. in Ω and

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,p,\Omega}\|u\|_{W^{k,p}(\Omega)}$$

The operator E is called a total extension for Ω.
You will have the opportunity to see how to construct such extension in a very specific case in Sheet 2.