# BO1 History of Mathematics
# Lecture IX
# Classical algebra: equation solving
# 1800BC – AD1800

MT 2021 Week 4

# Summary

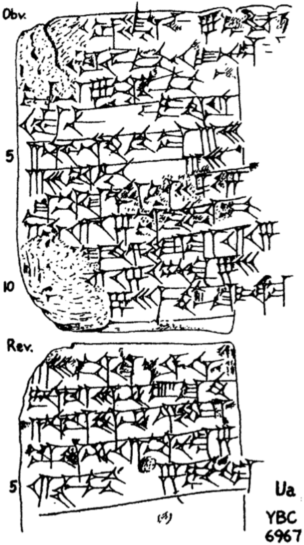
Part 1

- Early quadratic equations
- Cubic and quartic equations
- Further 16th-century developments

Part 2

- 17th century ideas
- 18th century ideas
- Looking back

# Completing the square, c. 1800 BC



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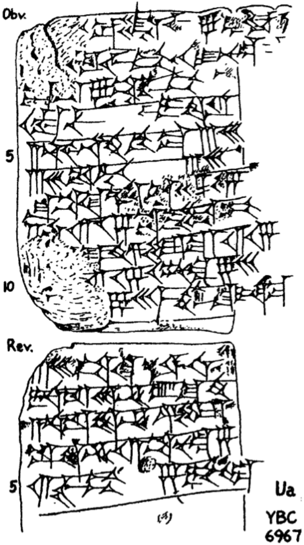


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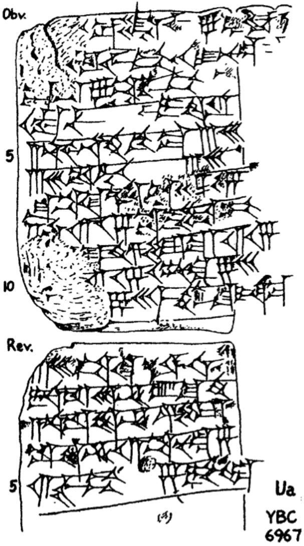


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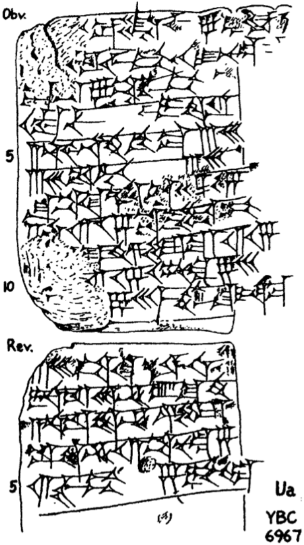


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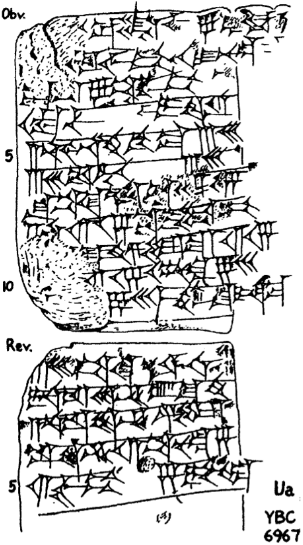


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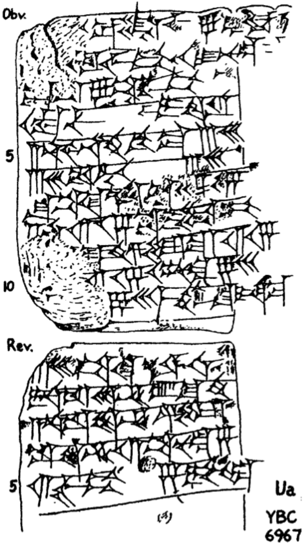


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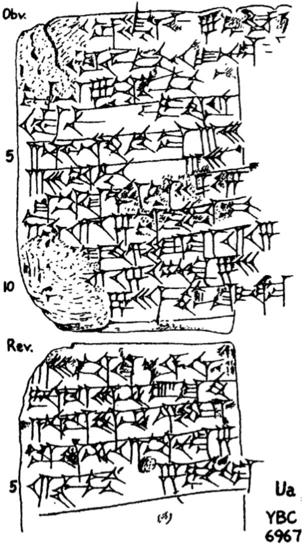


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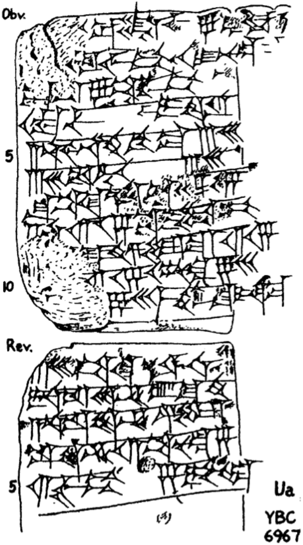


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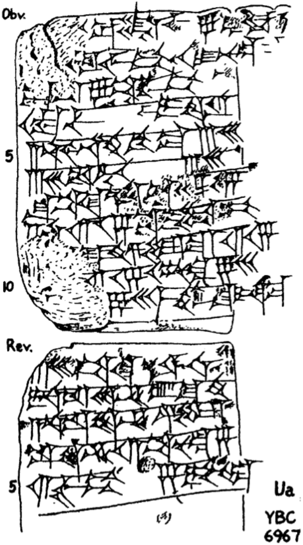


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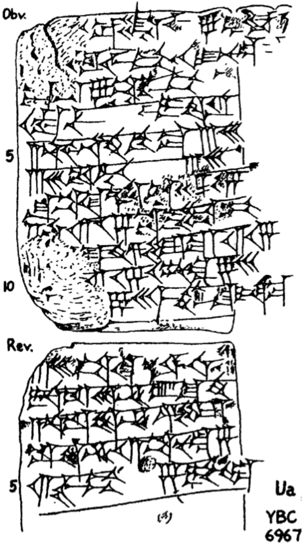


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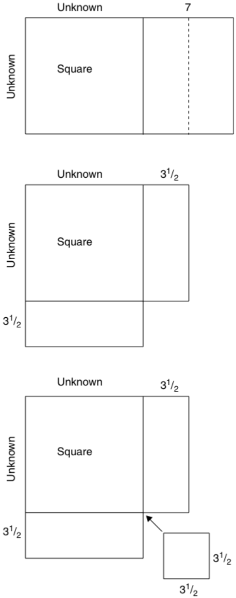


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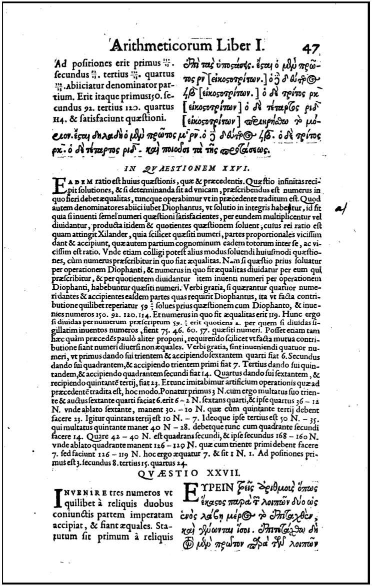
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution — but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

# Diophantus of Alexandria (3rd century AD)



Arithmeticorum Liber I. 47

Ad poſitiones erit primus $\frac{150}{23}$. ſecundus $\frac{92}{23}$. tertius $\frac{120}{23}$. quartus $\frac{114}{23}$. Abiiciatur denominator partium. Erit itaque primus 150. ſecundus 92. tertius 120. quartus 114. & ſatisfaciunt quæſtioni.

ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν πρῶτος ρν [εἰκοστοτρίτων.] ὁ δὲ δεύτερος ϟβ [εἰκοστοτρίτων.] ὁ δὲ τρίτος ρκ [εἰκοστοτρίτων] ὁ δὲ τέταρτος ριδ [εἰκοστοτρίτων] περιῃρήσθω τὸ μόριον. ἔσται δηλαδὴ ὁ μὲν πρῶτος μο ρν. ὁ δὲ δεύτερος ϟβ. ὁ δὲ τρίτος ρκ. ὁ δὲ τέταρτος ριδ. καὶ ποιοῦσι τὰ τῆς προτάσεως.

*IN QVAESTIONEM XXVI.*

EADEM ratio eſt huius quæſtionis, quæ & præcedentis. Quæſtio infinitas recipit ſolutiones, & ſi determinanda ſit ad vnicam, præſcribendus eſt numerus in quo fieri debet æqualitas, tuncque operabimur vt in præcedente traditum eſt. Quod autem denominatores abiici iubet Diophantus, vt ſolutio in integris habeatur, id fit quia ſi inuenti ſemel numeri quæſtioni ſatisfacientes, per eundem multiplicentur vel diuidantur, producta itidem & quotientes quæſtionem ſoluent, cuius rei ratio eſt quam attingit Xilander, quia ſcilicet quæſiti numeri, partes proportionales viciſſim dant & accipiunt, quæ autem partium cognominum eadem totorum inter ſe, ac viciſſim eſt ratio. Vnde etiam colligi poteſt alius modus ſoluendi huiuſmodi quæſtiones, cùm numerus præſcribitur in quo fiat æqualitas. Nam ſi quæſtio prius ſoluatur per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui præſcribitur, & per quotientem diuidantur item inuenti numeri per operationem Diophanti, habebuntur quæſiti numeri. Verbi gratia, ſi quærantur quatuor numeri dantes & accipientes eaſdem partes quas requirit Diophantus, ita vt facta contributione quilibet reperiatur $59\frac{1}{2}$ ſolues prius quæſtionem cum Diophanto, & inuenies numeros 150. 92. 120. 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo ſi diuidas per numerum præſcriptum $59.\frac{1}{2}$ erit quotiens 2. per quem ſi diuidas ſigillatim inuentos numeros, fient 75. 46. 60. 57. quæſiti numeri. Poſſet etiam tam hæc quàm præcedēs paulò aliter proponi, requirendo ſcilicet vt facta mutua contributione fiant numeri diuerſi non æquales. Verbi gratia, ſint inueniendi quatuor numeri, vt primus dando ſui trientem & accipiendo ſextantem quarti fiat 6. Secundus dando ſui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando ſui quintandem, & accipiendo quadrantem ſecundi fiat 14. Quartus dando ſui ſextantem, & recipiendo quintantē tertij, fiat 23. Et tunc imitabimur artificium operationis quæ ad præcedentē tradita eſt, hoc modo. Ponatur primus 3 N. cum ergo multatus ſuo triente & auctus ſextante quarti faciat 6. erit 6 – 2 N. ſextans quarti, & ipſe quartus 36 – 12 N. vnde ablato ſextante, manent 30. – 10 N. quæ cùm quintante tertij debent facere 23. Igitur quintans tertij eſt 10 N. – 7. Ideoque ipſe tertius eſt 50 N. – 35. qui multatus quintante manet 40 N – 28. debetque tunc cum quadrante ſecundi facere 14. Quare 42 – 40 N. eſt quadrans ſecundi, & ipſe ſecundus 168 – 160 N. vnde ablato quadrante manent 126 – 120 N. quæ cum triente primi debent facere 7. ſed faciunt 126 – 119 N. hoc ergo æquatur 7. & ſit 1 N. 1. Ad poſitiones primus eſt 3. ſecundus 8. tertius 15. quartus 24.

QVÆSTIO XXVII.

INVENIRE tres numeros vt quilibet à reliquis duobus coniunctis partem imperatam accipiat, & fiant æquales. Statutum ſit primum à reliquis

ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως ἕκαστος παρὰ τῶν λοιπῶν δύο ὡς ἑνὸς λάβῃ μέρος τὸ ἐπιταχθέν, καὶ γίνωνται ἴσοι. ἐπιτετάχθω δὴ τὸν μὲν πρῶτον παρὰ τῶν λοιπῶν

Problem I.27: Find two numbers such that their sum and product are given numbers

# Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:

1. Squares are equal to roots ($ax^2 = bx$)
2. Squares are equal to numbers ($ax^2 = c$)
3. Roots are equal to numbers ($bx = c$)
4. Squares and roots are equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers are equal to roots ($ax^2 + c = bx$)
6. Roots and numbers are equal to squares ($bx + c = ax^2$)

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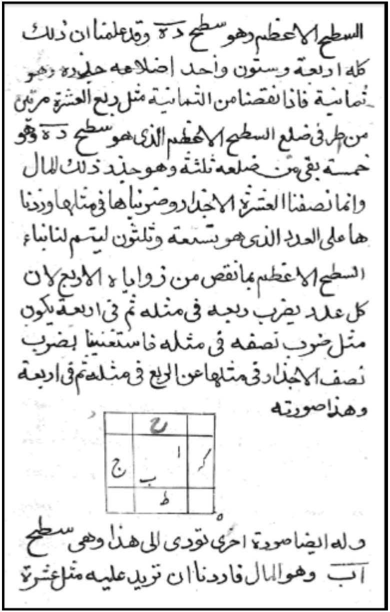


السطح الاعظم وهو سطح د ه وقد علمنا ان ذلك
كله اربعة وستون واحد اضلاعه جذره وهو
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين
من طرفي ضلع السطح الاعظم الذي هو سطح د ه فهو
خمسة بقي من ضلعه ثلثة وهو جذر ذلك المال
وانما نصفنا العشرة الاجذار وضربناها في مثلها وزدنا
ها على العدد الذي هو تسعة وثلثون ليتم لنا بناء
السطح الاعظم بما نقص من زواياه الاربع لان
كل عدد يضرب ربعه في مثله ثم في اربعة يكون
مثل ضرب نصفه في مثله فاستغنينا بضرب
نصف الاجذار في مثلها عن الربع في مثله ثم في اربعة
وهذه صورته

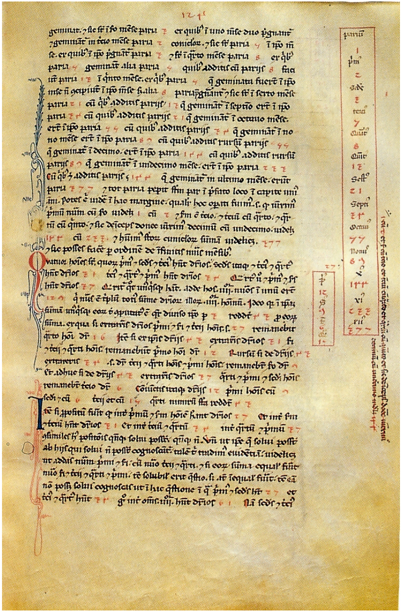
وله ايضا صورة اخرى تودي الى هذا وهي سطح
اب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on the previous slide

# Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

*Liber abaci* (or *Liber abbaci*), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



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- passed in rhyme to Girolamo Cardano (1539)

# Cubic equations (2)

$$x^3 + px = q$$

*When the cube with the things next after*
*Together equal some number apart*
*Find two others that by this differ*
*And this you will keep as a rule*
*That their product will always be equal*
*To a third cubed of the number of things*
*The difference then in general between*
*The sides of the cubes subtracted well*
*Will be your principal thing.*

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

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**NB:** In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

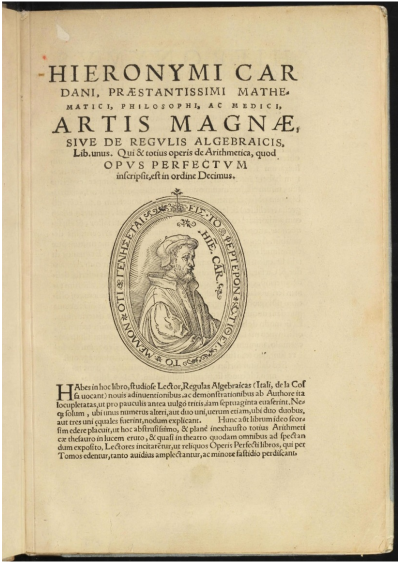
# Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

with similar expressions (in radicals) for the remaining two roots

# Cardano's *Ars magna, sive de regulis algebraicis* (1545)



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Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.



HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni ęquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

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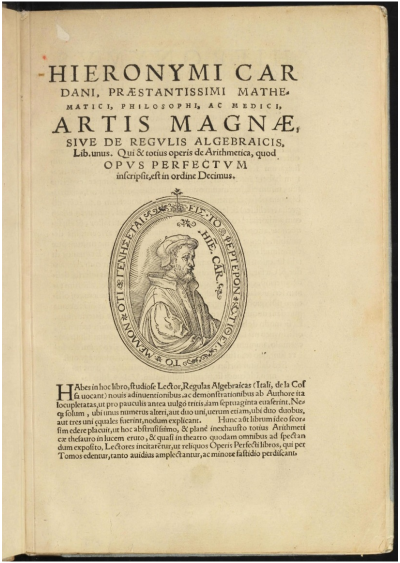

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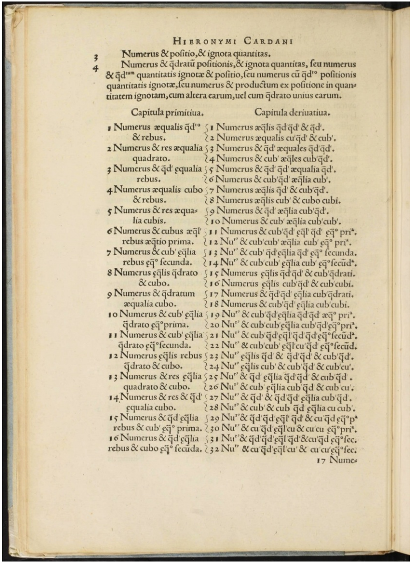
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HIE. CAR.

Habes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.



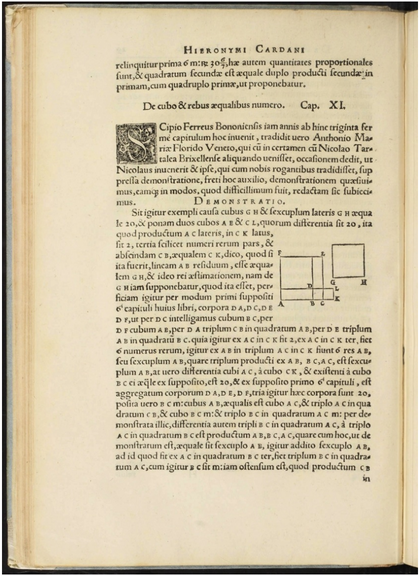
HIERONYMI CARDANI

3 Numerus & positio, & ignota quantitas.
4 Numerus & q̃dratũ positionis, & ignota quantitas, seu numerus & q̃d^tum quantitatis ignotæ & positio, seu numerus cũ q̃d^to positionis quantitatis ignotæ, seu numerus & productum ex positione in quantitatem ignotam, cum altera earum, uel cum q̃drato unius earum.

| Capitula primitiua. | Capitula deriuatiua. |
|---|---|
| 1 Numerus æqualis q̃d^to & rebus. | 1 Numerus æq̃lis q̃d'q̃d' & q̃d'. 2 Numerus æqualis cu'q̃d' & cub'. |
| 2 Numerus & res æqualia quadrato. | 3 Numerus & q̃d' æquales q̃d'q̃d'. 4 Numerus & cub' æq̃les cub'q̃d'. |
| 3 Numerus & q̃d' ęqualia rebus. | 5 Numerus & q̃d'q̃d' æqualia q̃d'. 6 Numerus & cub'q̃d' æq̃lia cub'. |
| 4 Numerus æqualis cubo & rebus. | 7 Numerus æq̃lis q̃d' & cub'q̃d'. 8 Numerus æq̃lis cub' & cubo cubi. |
| 5 Numerus & res æqualia cubis. | 9 Numerus & q̃d' æq̃lia cub'q̃d'. 10 Numerus & cub' æq̃lia cub'cub'. |
| 6 Numerus & cubus æq̃l' rebus æq̃tio prima. | 11 Numerus & cub'q̃d' ęq̃l' q̃d' ęq̃° pri^a. 12 Nu^rus & cub'cub' æq̃lia cub' ęq̃° pri^a. |
| 7 Numerus & cub' ęq̃lia rebus ęq̃° secunda. | 13 Nu^rus & cub' q̃d'ęq̃lia q̃d' ęq̃° secunda. 14 Nu^rus & cub'cub' ęq̃lia cub' ęq̃° secũd^a. |
| 8 Numerus ęq̃lis q̃drato & cubo. | 15 Numerus ęq̃lis q̃d'q̃d' & cub'q̃drati. 16 Numerus ęq̃lis cub'q̃d' & cub'cubi. |
| 9 Numerus & q̃dratum æqualia cubo. | 17 Numerus & q̃d'q̃d' ęq̃lia cub'q̃drati. 18 Numerus & cub'q̃d' ęq̃lia cub'cubi. |
| 10 Numerus & cub' ęq̃lia q̃drato ęq̃° prima. | 19 Nu^rus & cub'q̃d'ęq̃lia q̃d'q̃d' æq̃° pri^a. 20 Nu^rus & cub'cub'ęq̃lia cub'q̃d'ęq̃° pri^a. |
| 11 Numerus & cub' ęq̃lia q̃drato ęq̃° secunda. | 21 Nu^rus & cub'q̃d'ęq̃l'q̃d'q̃d'ęq̃° secũd^a. 22 Nu^rus & cub'cub' ęq̃l'cu'q̃d' ęq̃° secũd. |
| 12 Numerus ęq̃lis rebus q̃drato & cubo. | 23 Nu^rus ęq̃lis q̃d' & q̃d'q̃d' & cub'q̃d'. 24 Nu^rus ęq̃lis cub' & cub'q̃d' & cub'cu'. |
| 13 Numerus & res ęq̃lia quadrato & cubo. | 25 Nu^rus & q̃d' ęq̃lia q̃d'q̃d' & cub'q̃d'. 26 Nu^rus & cub' ęq̃lia cub'q̃d & cub'cu'. |
| 14 Numerus & res & q̃d' ęqualia cubo. | 27 Nu^rus & q̃d' & q̃d'q̃d' ęq̃lia cub'q̃d'. 28 Nu^rus & cub' & cub q̃d' ęq̃lia cu cub'. |
| 15 Numerus & q̃d ęq̃lia rebus & cub' ęq̃° prima. | 29 Nu^rus & q̃d'q̃d' ęq̃l' q̃d' & cu'q̃d ęq̃° p^a. 30 Nu^rus & cu'q̃d' ęq̃l'cu & cu'cu ęq̃° pri^a. |
| 16 Numerus & q̃d' ęq̃lia rebus & cubo ęq̃° secũda. | 31 Nu^rus & q̃d'q̃d' ęq̃l' q̃d' & cu'q̃d ęq̃° sec. 32 Nu^rus & cu'q̃d' ęq̃l'cu' & cu'cu' ęq̃° sec. |

17 Nume-

# Cardano on the cubic





relinquitur prima 6 m: ℞ 30$\frac{6}{7}$, hæc autem quantitates proportionales ſunt, & quadratum ſecundæ eſt æquale duplo producti ſecundæ in primam, cum quadruplo primæ, ut proponebatur.

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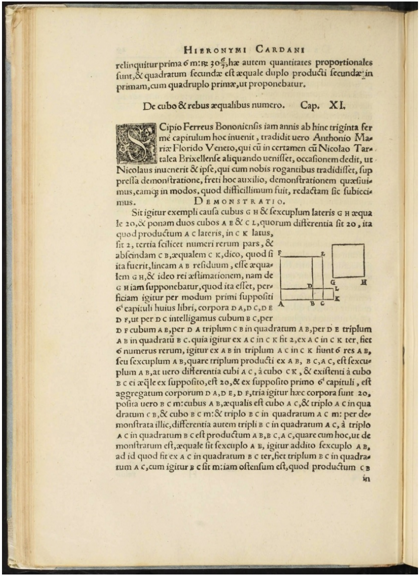
in

# Cardano on the cubic



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- Geometrical justification remains

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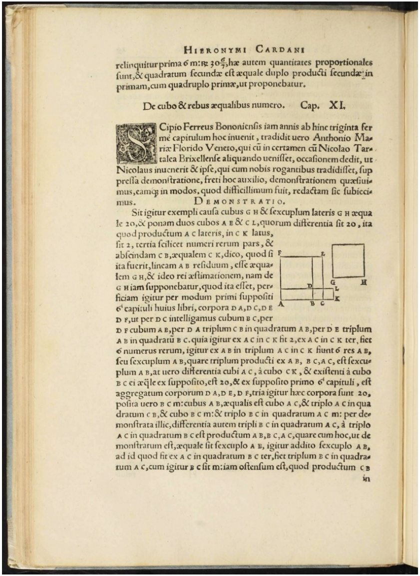

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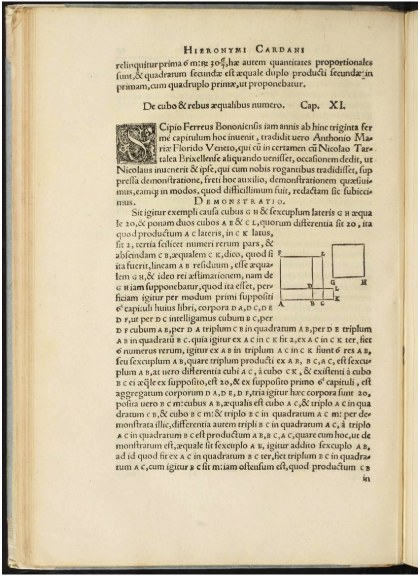

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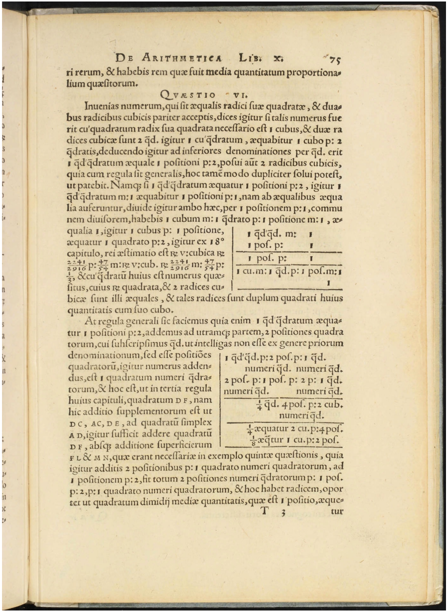
- Make substitution $x = y - \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

# Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic



DE ARITHMETICA LIB. X. 75

ri rerum, & habebis rem quæ fuit media quantitatum proportionalium quæſitorum.

QVÆSTIO VI.

Inuenias numerum, qui ſit æqualis radici ſuæ quadratæ, & duabus radicibus cubicis pariter acceptis, dices igitur ſi talis numerus fuerit cu'quadratum radix ſua quadrata neceſſario eſt 1 cubus, & duæ radices cubicæ ſunt 2 q̄d. igitur 1 cu'q̄dratum, æquabitur 1 cubo p: 2 q̄dratis, deducendo igitur ad inferiores denominationes per q̄d. erit 1 q̄d'q̄dratum æquale 1 poſitioni p:2, poſui aūt 2 radicibus cubicis, quia cum regula ſit generalis, hoc tamē modo dupliciter ſolui poteſt, ut patebit. Namq; ſi 1 q̄d'q̄dratum æquatur 1 poſitioni p:2, igitur 1 q̄d'q̄dratum m:1 æquabitur 1 poſitioni p:1, nam ab æqualibus æqualia auferuntur, diuide igitur ambo hæc, per 1 poſitionem p:1, communem diuiſorem, habebis 1 cubum m:1 q̄drato p:1 poſitione m:1, æqualia 1, igitur 1 cubus p: 1 poſitione, æquatur 1 quadrato p:2, igitur ex 18° capitulo, rei æſtimatio eſt R v:cubica R $\frac{2241}{2916}$ p: $\frac{47}{54}$ m: R v:cub. R $\frac{2241}{2916}$ m: $\frac{47}{54}$ p: $\frac{1}{3}$, & cu'q̄dratū huius eſt numerus quæſitus, cuius R quadrata, & 2 radices cubicæ ſunt illi æquales, & tales radices ſunt duplum quadrati huius quantitatis cum ſuo cubo.

1 q̄d'q̄d. m: 1
1 poſ. p: 1
1 poſ. p: 1
1 cu. m: 1 q̄d. p: 1 poſ. m: 1
1

At regula generali ſic faciemus quia enim 1 q̄d'q̄dratum æquatur 1 poſitioni p:2, addemus ad utramq; partem, 2 poſitiones quadratorum, cui ſubſcripſimus q̄d. ut intelligas non eſſe ex genere priorum denominationum, ſed eſſe poſitiões quadratorū, igitur numerus addendus, eſt 1 quadratum numeri q̄dratorum, & hoc eſt, ut in tertia regula huius capituli, quadratum D F, nam hic additio ſupplementorum eſt ut D C, A C, D E, ad quadratū ſimplex A D, igitur ſufficit addere quadratū D F, abſq; additione ſuperficierum F L & M N, quæ erant neceſſariæ in exemplo quintæ quæſtionis, quia igitur additis 2 poſitionibus p: 1 quadrato numeri quadratorum, ad 1 poſitionem p:2, fit totum 2 poſitiones numeri q̄dratorum p: 1 poſ. p:2, p:1 quadrato numeri quadratorum, & hoc habet radicem, oportet ut quadratum dimidij mediæ quantitatis, quæ eſt 1 poſitio, æquetur

1 q̄d'q̄d. p:2 poſ. p: 1 q̄d.
numeri q̄d. numeri q̄d.
2 poſ. p: 1 poſ. p: 2 p: 1 q̄d.
numeri q̄d. numeri q̄d.
$\frac{1}{4}$ q̄d. 4 poſ. p:2 cub.
numeri q̄d.
$\frac{1}{4}$ æquatur 2 cu. p:4 poſ.
$\frac{1}{8}$ æq̄tur 1 cu. p:2 poſ.

T 3 tur

# Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

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So the problem is reduced to solving a cubic equation and then a quadratic.

**NB:** In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

# Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's *Ars Magna* may also be found online here

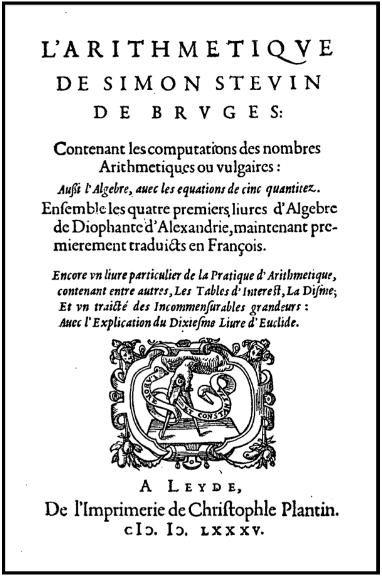
# Further 16th-century developments


Rafael Bombelli, *L'algebra* (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers

  [to be dealt with in a later lecture]

# Further 16th-century developments



L'ARITHMETIQVE
DE SIMON STEVIN
DE BRVGES:

Contenant les computations des nombres Arithmetiques ou vulgaires:

*Aussi l'Algebre, auec les equations de cinc quantitez.*

Ensemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.

*Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Interest, La Disme; Et vn traicté des Incommensurables grandeurs: Auec l'Explication du Dixiesme Liure d'Euclide.*



A LEYDE,
De l'Imprimerie de Christophle Plantin.
cIↃ. IↃ. LXXXV.

Simon Stevin, *L'arithmetique … aussi l'algebre* (1585):

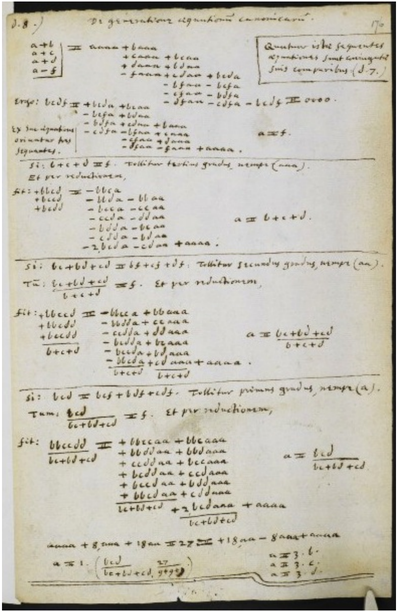
- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

# Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations

# Thomas Harriot (c. 1600)

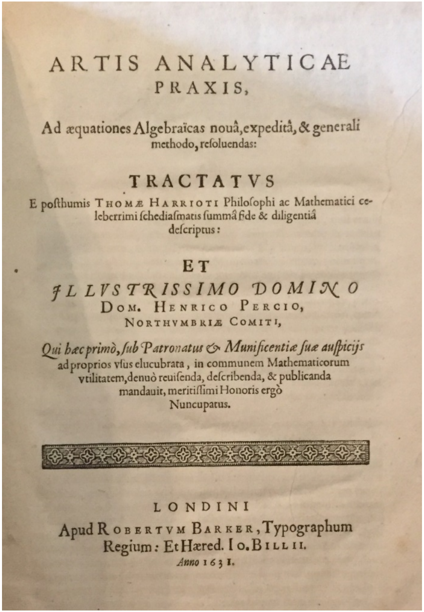


Add MS 6783 f. 176

Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

# Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, resoluendas:

TRACTATVS

E posthumis THOMÆ HARRIOTI Philosophi ac Mathematici celeberrimi schediasmatis summâ fide & diligentiâ descriptus:

ET

ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

*Qui hæc primò, sub Patronatus & Munificentiæ suæ auspicijs* ad proprios vsus elucubrata, in communem Mathematicorum vtilitatem, denuò reuisenda, describenda, & publicanda mandauit, meritissimi Honoris ergò Nuncupatus.

LONDINI
Apud ROBERTVM BARKER, Typographum Regium: Et Hæred. Io. BILLII.
*Anno* 1631.

Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

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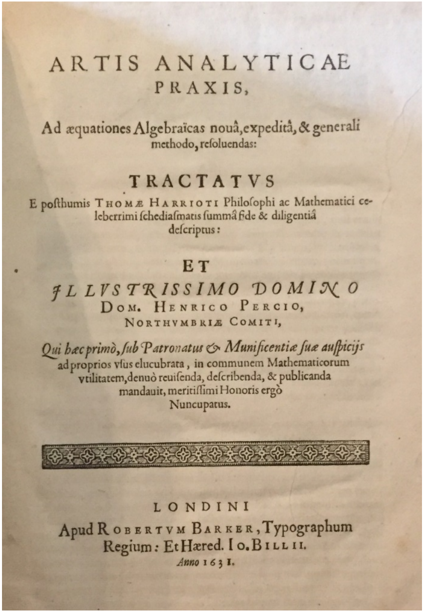

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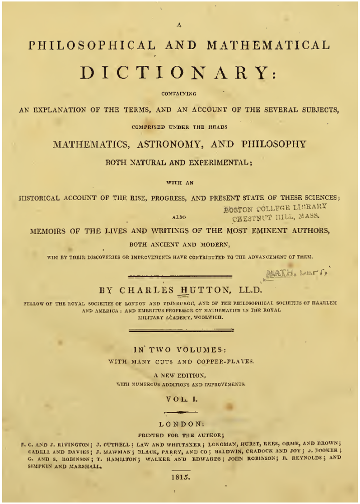
But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

# Commentary on Harriot



A

PHILOSOPHICAL AND MATHEMATICAL

DICTIONARY:

CONTAINING

AN EXPLANATION OF THE TERMS, AND AN ACCOUNT OF THE SEVERAL SUBJECTS,

COMPRISED UNDER THE HEADS

MATHEMATICS, ASTRONOMY, AND PHILOSOPHY

BOTH NATURAL AND EXPERIMENTAL;

WITH AN

HISTORICAL ACCOUNT OF THE RISE, PROGRESS, AND PRESENT STATE OF THESE SCIENCES;



ALSO

MEMOIRS OF THE LIVES AND WRITINGS OF THE MOST EMINENT AUTHORS,

BOTH ANCIENT AND MODERN,

WHO BY THEIR DISCOVERIES OR IMPROVEMENTS HAVE CONTRIBUTED TO THE ADVANCEMENT OF THEM.



BY CHARLES HUTTON, LL.D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH, AND OF THE PHILOSOPHICAL SOCIETIES OF HAARLEM AND AMERICA; AND EMERITUS PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

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A NEW EDITION,

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1815.

Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

*He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; . . .*

# Part 2: Theory of Equations

# Algebra in the 17th century

From 1600 onwards, ‘algebra’ as a set of recipes and techniques began to diverge in two (linked) directions:

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- ‘algebra’ as a tool or a language (a.k.a. ‘analysis’ or the ‘analytic art’)
- ‘algebra’ as an object of study in its own right (the ‘theory of equations’)

# Descartes on algebra

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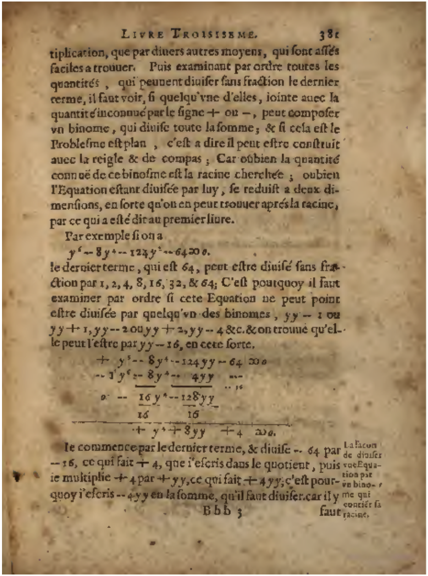
  has at most 3 positive roots and at most one negative;
- can always make a transformation to remove the second-highest term.

# Descartes on cubics




tiplication, que par diuers autres moyens, qui sont assés faciles a trouuer. Puis examinant par ordre toutes les quantités, qui peuuent diuiser sans fraction le dernier terme, il faut voir, si quelqu'vne d'elles, iointe auec la quantité inconnuë par le signe + ou −, peut composer vn binome, qui diuise toute la somme; & si cela est le Problesme est plan, c'est a dire il peut estre construit auec la reigle & le compas; Car oubien la quantité connuë de ce binosme est la racine cherchée; oubien l'Equation estant diuisée par luy, se reduist a deux dimensions, en sorte qu'on en peut trouuer aprés la racine, par ce qui a esté dit au premier liure.

Par exemple si on a

$y^6 - 8y^4 - 124y^2 - 64 \infty 0.$

le dernier terme, qui est 64, peut estre diuisé sans fraction par 1, 2, 4, 8, 16, 32, & 64; C'est pourquoy il faut examiner par ordre si cete Equation ne peut point estre diuisée par quelqu'vn des binomes, $yy - 1$ ou $yy + 1$, $yy - 2$ ou $yy + 2$, $yy - 4$ &c. & on trouue qu'elle peut l'estre par $yy - 16$, en cete sorte.

$+ y^6 - 8y^4 - 124yy - 64 \infty 0$
$- 1y^6 - 8y^4 - 4yy \quad - 16$
$0 \quad - 16y^4 - 128yy$
$16 \quad 16$
$+ y^4 + 8yy + 4 \infty 0.$

Ie commence par le dernier terme, & diuise −64 par −16, ce qui fait +4, que i'escris dans le quotient, puis ie multiplie +4 par +yy, ce qui fait +4yy; c'est pourquoy i'escris −4yy en la somme, qu'il faut diuiser. car il y faut

La façon de diuiser vne Equation par vn binome qui contient sa racine.



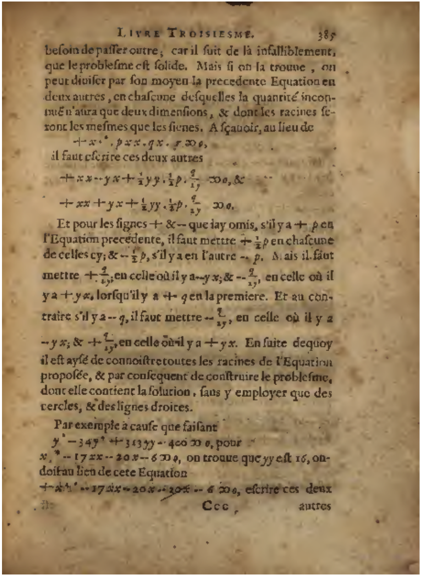
Search for roots of a cubic by examining the factors of the constant term:

if $\alpha$ is such a factor, test whether $x - \alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

# Descartes on quartics





besoin de passer outre; car il suit de là infalliblement, que le problesme est solide. Mais si on la trouue, on peut diuiser par son moyen la precedente Equation en deux autres, en chascune desquelles la quantité inconnuë n'aura que deux dimensions, & dont les racines seront les mesmes que les siennes. A sçauoir, au lieu de

$+x^4 * .pxx.qx.r \infty 0$,

il faut escrire ces deux autres

$+xx - yx + \frac{1}{2}yy . \frac{1}{2}p . \frac{q}{2y} \infty 0$, &

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Par exemple à cause que faisant

$y^6 - 34y^4 + 313yy - 400 \infty 0$, pour

$x^4 * -17xx - 20x - 6 \infty 0$, on trouue que $yy$ est 16, on doit au lieu de cete Equation

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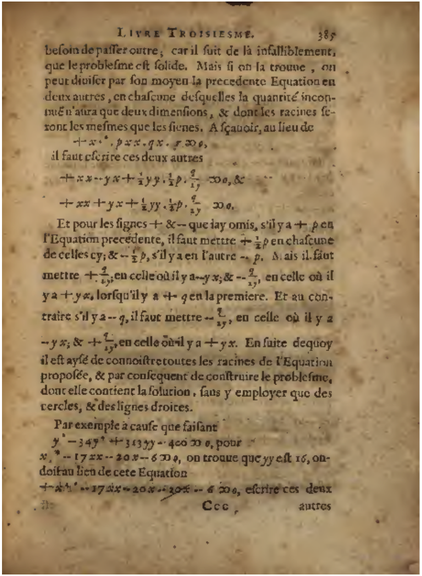



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$$+x^4 * .pxx.qx.r \propto 0,$$

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$$+xx - yx + \tfrac{1}{2}yy.\tfrac{1}{2}p.\tfrac{q}{2y} \propto 0, \&$$

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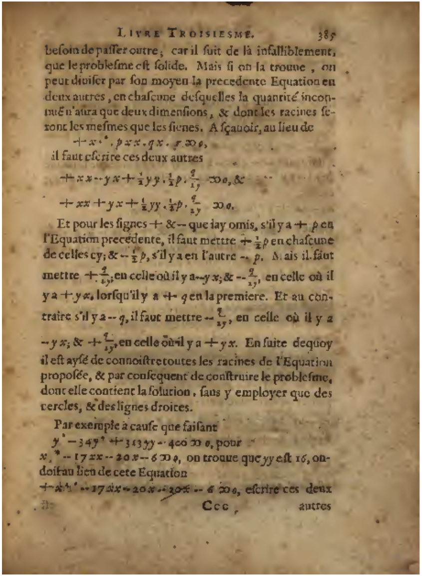

LIVRE TROISIESME. 385

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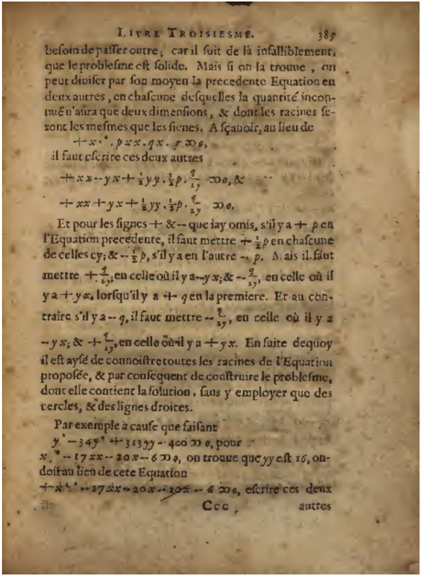

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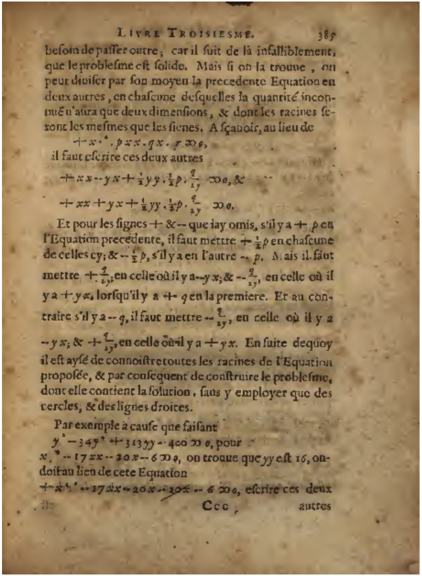
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As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

# Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via $+$, $-$, $\times$, $\div$, $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, $\ldots$

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So did anything interesting happen in algebra during the 17th and 18th centuries?

# A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

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Fair point? Or not?

# Some 17th-century developments: Hudde's rule (1657)

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quæro, per Methodum ſuperiùs explicatam, maximum earum communem diviſorem; atque hujus ope æquationem Propoſitam toties divido, quoties id fieri poteſt.

Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \infty 0$, in qua duæ ſunt æquales radices. Multiplico ergo ipſam per Arithmeticam Progreſſionem qualemcunque, hoc eſt, cujus incrementum vel decrementum ſit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus ſit vel 0, vel +, vel − quam 0: Ita ut ſemper ejus ope talis terminus æquationis tolli poſſit, qualem quis voluerit, collocando tantùm ſub eo 0.

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Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$

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Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \infty 0$.

Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \infty 0$

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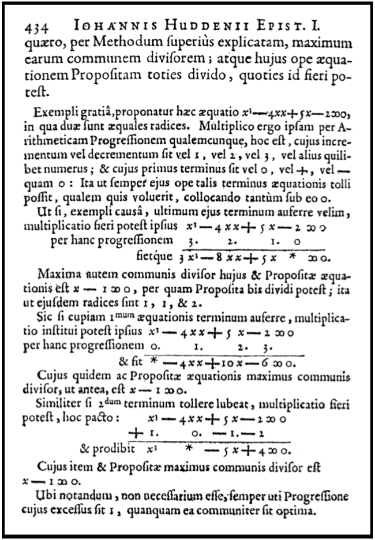

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as does $-4xx + 10x - 6 = 0$.

(Modern form of rule: if $r$ is a double root of $f(x) = 0$, then it is a root of $f'(x) = 0$ also.)

See *Mathematics emerging*, §12.2.2.

# Some 17th-century developments: Tschirnhaus transformations (1683)



## METHODUS AUFERENDI OMNES TER*minos intermedios ex data æquatione,*

*per D. T.*

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Primo itaque loco, ad hoc attendendum; ſit data aliqua æquatio cubica $x^3 - pxx + qx - r = 0$, in qua x radices hujus æquationis deſignat; p, q, r, cognitas quantitates repræſentant: ad auferendum jam ſecundum terminum ſupponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas x abſit, & erit

$$\begin{array}{llll} y^3 & +3ayy & +3aay & +a^3 = 0 \\ & -pyy & -2pay & -paa \\ & & +qy & +qa \\ & & & -r \end{array}$$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = 0$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

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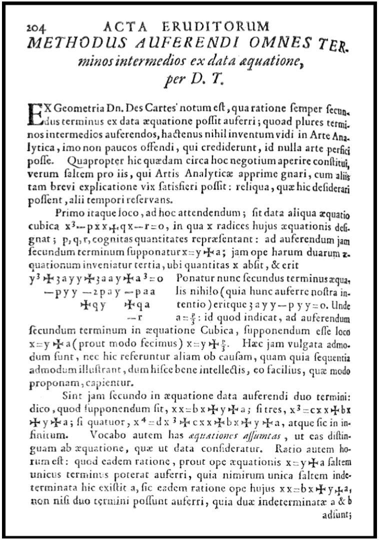

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For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)

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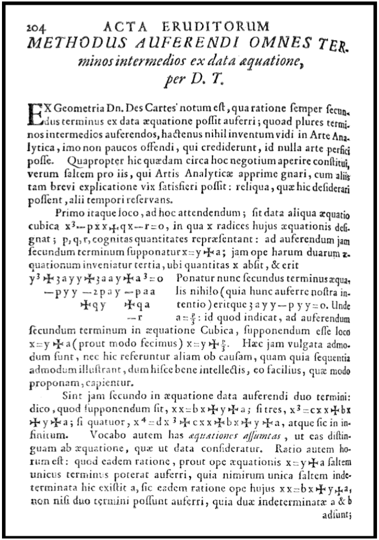

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$$\begin{array}{llll} y^3 + 3ayy & + 3aay & + a^3 & = 0 \\ \phantom{y^3} - pyy & - 2pay & - paa & \\ & + qy & + qa & \\ & & - r & \end{array}$$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = 0$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

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For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?

# Some 17th-century developments: Tschirnhaus transformations (1683)



*METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione,*
*per D. T.*

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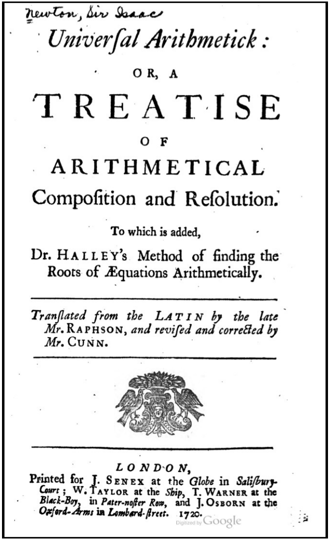
See *Mathematics emerging*, §12.2.3.

# An 18th-century development: Newton's *Arithmetica universalis* (1707)



Newton, Sir Isaac

*Univerſal Arithmetick:*

OR, A

TREATISE

OF

ARITHMETICAL

Compoſition and Reſolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

*Tranſlated from the* LATIN *by the late Mr.* RAPHSON, *and reviſed and corrected by Mr.* CUNN.

*LONDON,*

Printed for J. SENEX at the *Globe* in *Saliſbury-Court*; W. TAYLOR at the *Ship*, T. WARNER at the *Black-Boy*, in *Pater-noſter Row*, and J. OSBORN at the *Oxford-Arms* in *Lombard-ſtreet*. 1720.



Rules for sums of powers of roots of

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \cdots = 0$$

$$\begin{aligned}
\text{sum of roots} &= p \\
\text{sum of roots}^2 &= pa - 2q \\
\text{sum of roots}^3 &= pb - qa + 3r \\
\text{sum of roots}^4 &= pc - qb + ra - 4s
\end{aligned}$$

# Developments of the 17th and 18th centuries

- Symbolic notation

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- ... of how to manipulate them
- ... of how to solve them numerically
- The leaving behind of geometric intuition?

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Recall:

- cubic equations can be solved by means of quadratics

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- though always reducible to $(n-1)!$
- possibly further reducible to $(n-2)!$

# Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

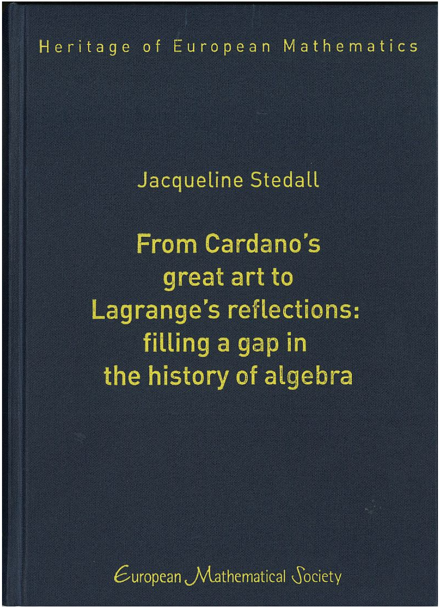
seeking to identify a uniform method that could be extended to higher degree

# A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

# Filling a gap in the history of algebra (2011)




*The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations . . .* [Preface]

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> *This assertion . . . from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'.*

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