BO1 History of Mathematics
Lecture VI
Infinite series
Part 3: The 18th century

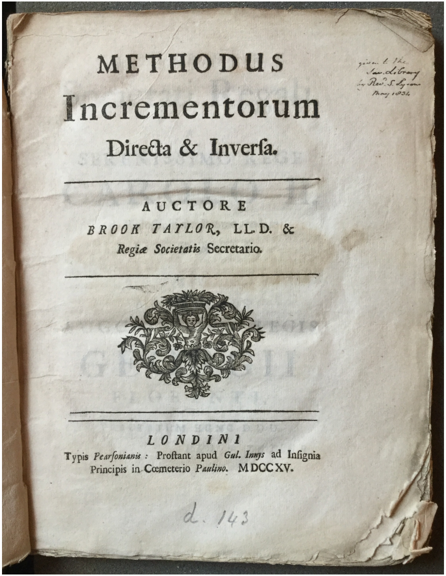
MT 2021 Week 3

Move on to the 18th century

Eighteenth century:

- as in 17th century, much progress;
- also many questions and doubts

Taylor series



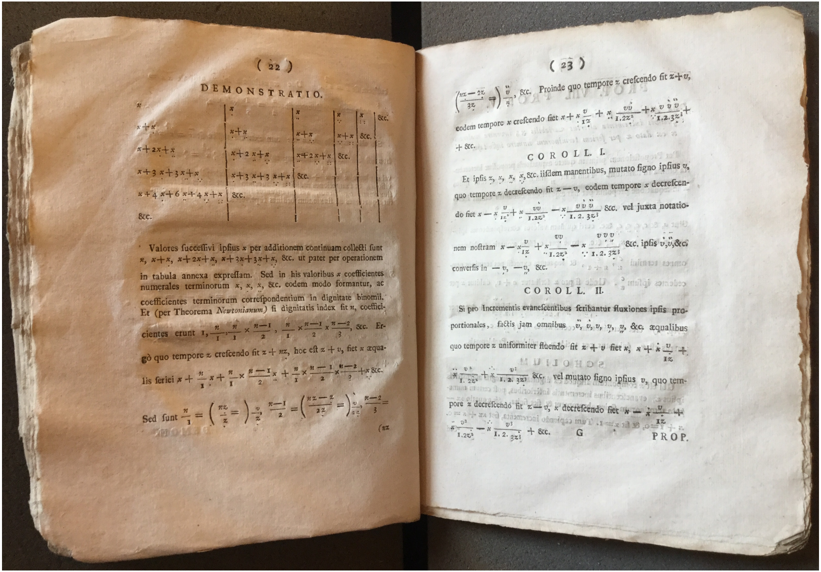
METHODUS
Incrementorum
Directa & Inversa.

AUCTORE
BROOK TAYLOR, LL.D. &
Regiæ Societatis Secretario.

LONDINI
Typis *Pearsonianis*: Prostant apud *Gul. Innys* ad Insignia
Principis in Cœmeterio *Paulino*. MDCCXV.

Brook Taylor,
The method of direct and inverse increments (1715)

Taylor series


(22)

DEMONSTRATIO.

Valores fucceffivi ipfius x per additionem continuam collecti funt x, $x+\dot{x}$, $x+2\dot{x}+\ddot{x}$, $x+3\dot{x}+3\ddot{x}+\dddot{x}$, &c. ut patet per operationem in tabula annexa expreffam. Sed in his valoribus x coefficientes numerales terminorum x, $\dot{x}$, $\ddot{x}$, &c. eodem modo formantur, ac coefficientes terminorum correfpondentium in dignitate binomii. Et (per Theorema *Newtonianum*) fi dignitatis index fit n, coefficientes erunt $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. Ergò quo tempore z crefcendo fit $z+nz$, hoc eft $z+v$, fiet x æqualis feriei $x+\frac{n}{1}\dot{x}+\frac{n}{1}\times\frac{n-1}{2}\ddot{x}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\dddot{x}+$ &c.

Sed funt $\frac{n}{1}=\left(\frac{nz}{z}=\right)\frac{v}{z}$, $\frac{n-1}{2}=\left(\frac{nz-z}{2z}=\right)\ldots$, $\frac{n-2}{3}=$

(23)

&c. Proinde quo tempore z crefcendo fit $z+v$, eodem tempore x crefcendo fiet $x+\dot{x}\frac{v}{1z}+\ddot{x}\frac{vv}{1.2z^2}+\dddot{x}\frac{vvv}{1.2.3z^3}+$ &c.

COROLL. I.

Et ipfis z, x, $\dot{x}$, $\ddot{x}$, &c. iifdem manentibus, mutato figno ipfius v, quo tempore z decrefcendo fit $z-v$, eodem tempore x decrefcendo fiet $x-\dot{x}\frac{v}{1z}+\ddot{x}\frac{vv}{1.2z^2}-\dddot{x}\frac{vvv}{1.2.3z^3}$ &c. vel juxta notationem noftram $x-\dot{x}\frac{v}{1z}+\ddot{x}\frac{vv}{1.2z^2}-\dddot{x}\frac{vvv}{1.2.3z^3}$ &c. ipfis v, v, &c. converfis in $-v$, $-v$, &c.

COROLL. II.

Si pro Incrementis evanefcentibus fcribantur fluxiones ipfis proportionales, factis jam omnibus v, v, v, v, v, &c. æqualibus quo tempore z uniformiter fluendo fit $z+v$ fiet x, $x+\dot{x}\frac{v}{1z}+\ddot{x}\frac{v^2}{1.2z^2}+\dddot{x}\frac{v^3}{1.2.3z^3}$ &c. vel mutato figno ipfius v, quo tempore z decrefcendo fit $z-v$, x decrefcendo fiet $x-\dot{x}\frac{v}{1z}+\ddot{x}\frac{v^2}{1.2z^2}-\dddot{x}\frac{v^3}{1.2.3z^3}+$ &c.

G PROP.

(See: *Mathematics emerging*, §8.2.1.)

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx),

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$),

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt;

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt; after a further interval of δt, x has become $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$;

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt; after a further interval of δt, x has become $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$; continuing:

$$x + \frac{n}{1}\delta x + \frac{n(n-1)}{1 \cdot 2}\delta(\delta x) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\delta(\delta(\delta x)) + \cdots$$

Taylor series

Taylor denoted a small change in x by $\underset{\cdot}{x}$ (our δx), a small change in $\underset{\cdot}{x}$ by $\underset{\cdot\cdot}{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt; after a further interval of δt, x has become $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$; continuing:

$$x + \frac{n}{1}\delta x + \frac{n(n-1)}{1 \cdot 2}\delta(\delta x) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\delta(\delta(\delta x)) + \cdots$$

$$= x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1\cdot 2\cdot(\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1\cdot 2\cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant
- $\delta x \propto \dot{x}$ and $\delta z \propto \dot{z}$, so in each case the former can be replaced by the latter

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1\cdot 2\cdot(\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1\cdot 2\cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant
- $\delta x \propto \dot{x}$ and $\delta z \propto \dot{z}$, so in each case the former can be replaced by the latter

In essence (in modern terms): $\frac{\delta x}{\delta z} \to \frac{dx}{dz}$, $\frac{\delta(\delta x)}{(\delta z)^2} \to \frac{d^2x}{dz^2}$, and so on

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1\cdot 2\cdot(\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1\cdot 2\cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant
- $\delta x \propto \dot{x}$ and $\delta z \propto \dot{z}$, so in each case the former can be replaced by the latter

In essence (in modern terms): $\dfrac{\delta x}{\delta z} \to \dfrac{dx}{dz}$, $\dfrac{\delta(\delta x)}{(\delta z)^2} \to \dfrac{d^2x}{dz^2}$, and so on

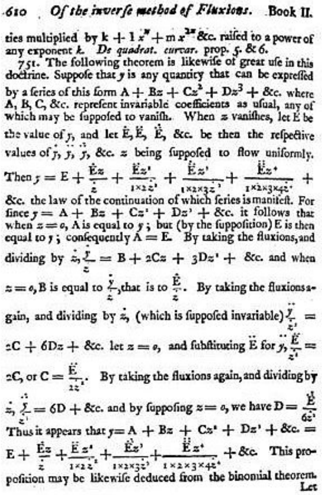
Again in modern terms, we arrive at:

$$x + \frac{dx}{dz}v + \frac{d^2x}{dz^2}\frac{v^2}{1\cdot 2} + \frac{d^3x}{dz^3}\frac{v^3}{1\cdot 2\cdot 3} + \cdots$$

Cf. Taylor's notation in *Mathematics Emerging*, §8.1.2

Maclaurin's *Treatise of fluxions*, vol. II, p. 610

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$



ties multiplied by $k + lx^n + mx^{2n}$ &c. raiſed to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewiſe of great uſe in this doctrine. Suppoſe that y is any quantity that can be expreſſed by a ſeries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. repreſent invariable coefficients as uſual, any of which may be ſuppoſed to vaniſh. When z vaniſhes, let E be the value of y, and let $\dot{E}, \ddot{E}, \dddot{E}$, &c. be then the reſpective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being ſuppoſed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. the law of the continuation of which ſeries is manifeſt. For ſince $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = 0$, A is equal to y; but (by the ſuppoſition) E is then equal to y; conſequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = 0$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is ſuppoſed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = 0$, and ſubſtituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by ſuppoſing $z = 0$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. This propoſition may be likewiſe deduced from the binomial theorem.

Let

Maclaurin's *Treatise of fluxions*, vol. II, p. 610

ties multiplied by $k + l x^n + m x^{2n}$ &c. raiſed to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

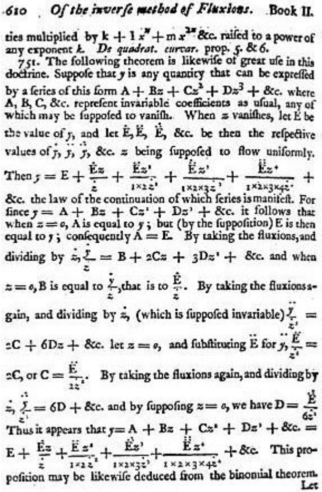
751. The following theorem is likewiſe of great uſe in this doctrine. Suppoſe that y is any quantity that can be expreſſed by a ſeries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. repreſent invariable coefficients as uſual, any of which may be ſuppoſed to vaniſh. When z vaniſhes, let E be the value of y, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the reſpective values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. z being ſuppoſed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. the law of the continuation of which ſeries is manifeſt. For ſince $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = 0$, A is equal to y; but (by the ſuppoſition) E is then equal to y; conſequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = 0$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is ſuppoſed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = 0$, and ſubſtituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by ſuppoſing $z = 0$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. This propoſition may be likewiſe deduced from the binomial theorem.

Let

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

Maclaurin's *Treatise of fluxions*, vol. II, p. 610



610 *Of the inverse method of Fluxions.* Book II.

ties multiplied by $k + l x^n + m x^{2n}$ &c. raised to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewise of great use in this doctrine. Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. represent invariable coefficients as usual, any of which may be supposed to vanish. When z vanishes, let E be the value of y, and let $\dot{E}, \ddot{E}, \dddot{E}$, &c. be then the respective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being supposed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. the law of the continuation of which series is manifest. For since $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = o$, A is equal to y; but (by the supposition) E is then equal to y; consequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = o$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is supposed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = o$, and substituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by supposing $z = o$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. This proposition may be likewise deduced from the binomial theorem.

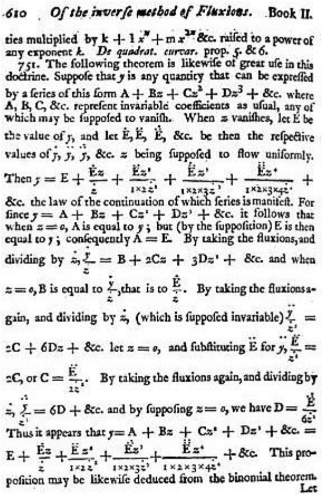
Let

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

z is assumed to flow uniformly, so that $\dot{z} = \text{const}$

Maclaurin's *Treatise of fluxions*, vol. II, p. 610



610 *Of the inverse method of Fluxions.* Book II.

ties multiplied by $k + l x^n + m x^{2n}$ &c. raised to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewise of great use in this doctrine. Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. represent invariable coefficients as usual, any of which may be supposed to vanish. When z vanishes, let E be the value of y, and let $\dot{E}, \ddot{E}, \dddot{E}$ &c. be then the respective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being supposed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^4} +$ &c. the law of the continuation of which series is manifest. For since $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = 0$, A is equal to y; but (by the supposition) E is then equal to y; consequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = 0$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is supposed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = 0$, and substituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by supposing $z = 0$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^4} +$ &c. This proposition may be likewise deduced from the binomial theorem.

Let

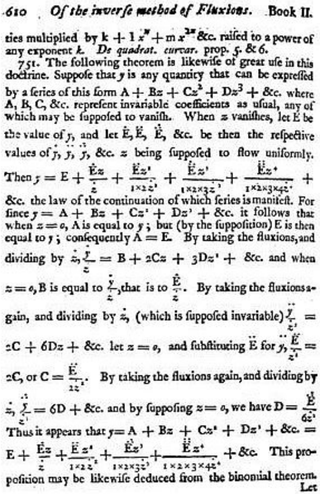
Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

z is assumed to flow uniformly, so that $\dot{z} = \text{const}$

By repeatedly taking fluxions, we may calculate in turn: $A = E$, $B = \frac{\dot{E}}{\dot{z}}$, $C = \frac{\ddot{E}}{2\dot{z}^2}$, $D = \frac{\dddot{E}}{6\dot{z}^3}$, etc.

Maclaurin's *Treatise of fluxions*, vol. II, p. 610



610 *Of the inverſe method of Fluxions.* Book II.

ties multiplied by $k + l x^n + m x^{2n}$ &c. raiſed to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewiſe of great uſe in this doctrine. Suppoſe that y is any quantity that can be expreſſed by a ſeries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. repreſent invariable coefficients as uſual, any of which may be ſuppoſed to vaniſh. When z vaniſhes, let E be the value of y, and let $\dot{E}, \ddot{E}, \dddot{E}$, &c. be then the reſpective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being ſuppoſed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^4} +$ &c. the law of the continuation of which ſeries is manifeſt. For ſince $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = 0$, A is equal to y; but (by the ſuppoſition) E is then equal to y; conſequently A = E. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = 0$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is ſuppoſed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = 0$, and ſubſtituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by ſuppoſing $z = 0$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^4} +$ &c. This propoſition may be likewiſe deduced from the binomial theorem.

Let

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

z is assumed to flow uniformly, so that $\dot{z} = \text{const}$

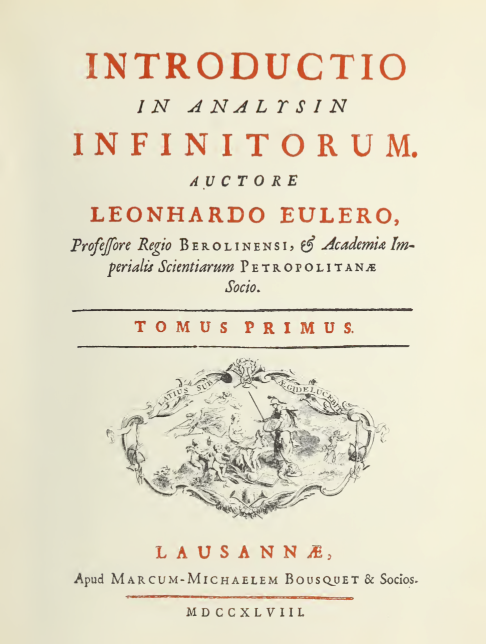
By repeatedly taking fluxions, we may calculate in turn: $A = E$, $B = \frac{\dot{E}}{\dot{z}}$, $C = \frac{\ddot{E}}{2\dot{z}^2}$, $D = \frac{\dddot{E}}{6\dot{z}^3}$, etc.

"the law of the continuation of [the] series is manifest"

(*Mathematics emerging*, §8.2.2.)

Euler's *Introductio*

Leonhard Euler, *Introduction to analysis of the infinite* (1748)



INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

Euler's *Introductio*

Incorporated power series into the definition of a function.

Euler's *Introductio*

Incorporated power series into the definition of a function.

> *Since fractional or irrational functions of z are not confined to complete forms* $A + Bz + Cz^2 + Dz^3 +$ *etc. where the number of terms is finite, it is usual to seek expressions of this kind carrying on to infinity, which exhibit the value of the function whether fractional or irrational. And indeed the nature of transcendental functions is thought to be better understood if expressed in this kind of form, even though infinite.*

Euler's *Introductio*

Incorporated power series into the definition of a function.

> *Since fractional or irrational functions of z are not confined to complete forms* $A + Bz + Cz^2 + Dz^3 +$ *etc. where the number of terms is finite, it is usual to seek expressions of this kind carrying on to infinity, which exhibit the value of the function whether fractional or irrational. And indeed the nature of transcendental functions is thought to be better understood if expressed in this kind of form, even though infinite.*

Euler derived series for sine, cosine, exp, log, etc.;

Euler's *Introductio*

Incorporated power series into the definition of a function.

> *Since fractional or irrational functions of z are not confined to complete forms* $A + Bz + Cz^2 + Dz^3 +$ *etc. where the number of terms is finite, it is usual to seek expressions of this kind carrying on to infinity, which exhibit the value of the function whether fractional or irrational. And indeed the nature of transcendental functions is thought to be better understood if expressed in this kind of form, even though infinite.*

Euler derived series for sine, cosine, exp, log, etc.;

he also discovered relationships between them, for example:

$$\cos v = \frac{1}{2}(e^{iv} + e^{-iv})$$

Doubts

XXXV^ME^ MÉMOIRE.

Réflexions ſur les Suites & ſur les Racines imaginaires.

§. I.

Réflexions ſur les ſuites divergentes ou convergentes.

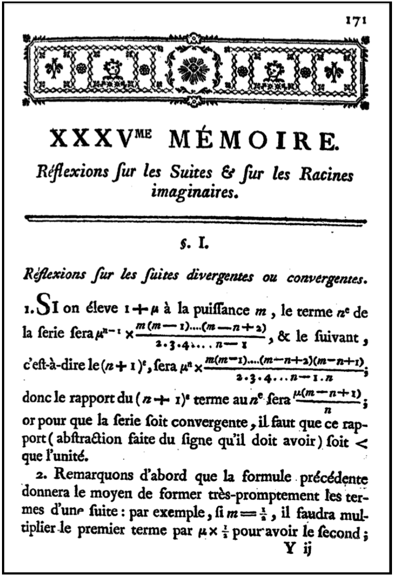
1. SI on éleve $1+\mu$ à la puiſſance m, le terme n^e de la ſerie ſera $\mu^{n-1} \times \frac{m(m-1)\ldots(m-n+2)}{2.3.4\ldots n-1}$, & le ſuivant, c'eſt-à-dire le $(n+1)^e$, ſera $\mu^n \times \frac{m(m-1)\ldots(m-n+2)(m-n+1)}{2.3.4\ldots n-1.n}$; donc le rapport du $(n+1)^e$ terme au n^e ſera $\frac{\mu(m-n+1)}{n}$; or pour que la ſerie ſoit convergente, il faut que ce rapport (abſtraction faite du ſigne qu'il doit avoir) ſoit $<$ que l'unité.

2. Remarquons d'abord que la formule précédente donnera le moyen de former très-promptement les termes d'une ſuite: par exemple, ſi $m = \frac{1}{2}$, il faudra multiplier le premier terme par $\mu \times \frac{1}{2}$ pour avoir le ſecond;

D'Alembert, 1761:

... all reasoning and calculation based on series that do not converge, or that one may suppose not to, always seems to me extremely suspect, even when the results of this reasoning agree with truths known in other ways.

Doubts



XXXV^ME MÉMOIRE.

Réflexions sur les Suites & sur les Racines imaginaires.

§. I.

Réflexions sur les suites divergentes ou convergentes.

1. Si on éleve $1+\mu$ à la puissance m, le terme n^e de la serie sera $\mu^{n-1} \times \frac{m(m-1)\ldots(m-n+2)}{2.3.4\ldots n-1}$, & le suivant, c'est-à-dire le $(n+1)^e$, sera $\mu^n \times \frac{m(m-1)\ldots(m-n+2)(m-n+1)}{2.3.4\ldots n-1.n}$; donc le rapport du $(n+1)^e$ terme au n^e sera $\frac{\mu(m-n+1)}{n}$; or pour que la serie soit convergente, il faut que ce rapport (abstraction faite du signe qu'il doit avoir) soit $<$ que l'unité.

2. Remarquons d'abord que la formule précédente donnera le moyen de former très-promptement les termes d'une suite : par exemple, si $m = \frac{1}{2}$, il faudra multiplier le premier terme par $\mu \times \frac{1}{2}$ pour avoir le second;

Y ij

D'Alembert, 1761:

... all reasoning and calculation based on series that do not converge, or that one may suppose not to, always seems to me extremely suspect, even when the results of this reasoning agree with truths known in other ways.

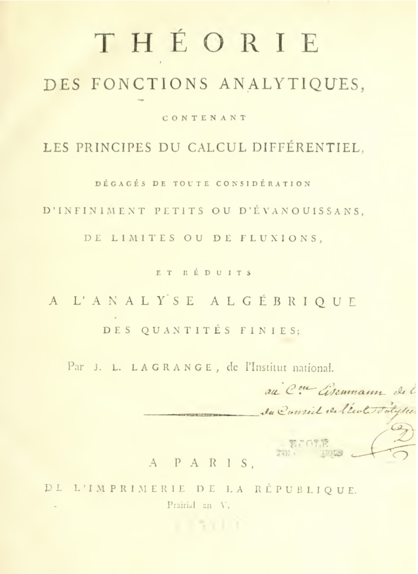
Introduced, without proof, what came to be known (in a more general setting) as d'Alembert's ratio test.

(See: *Mathematics emerging*, §8.3.1.)

Lagrange's use of series

J.-L. Lagrange, *Théorie des fonctions analytiques* (1797)

Lagrange's use of series: an attempt to liberate calculus from infinitely small quantities (essentially by treating only those functions that may be described by power series)



THÉORIE

DES FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION

D'INFINIMENT PETITS OU D'ÉVANOUISSANS,

DE LIMITES OU DE FLUXIONS,

ET RÉDUITS

A L'ANALYSE ALGÉBRIQUE

DES QUANTITÉS FINIES;

Par J. L. LAGRANGE, de l'Institut national.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

Prairial an V.

Lagrange and convergence

> *... [one needs] a way of stopping the expansion of the series at any term one wants and of estimating the value of the remainder of the series.*
>
> *This problem, one of the most important in the theory of series, has not yet been resolved in a general way*

Lagrange found bounds for the ‘remainder’ ...

and applied his findings to the binomial series ...

thus proving what Newton had taken for granted

(See: *Mathematics emerging*, §8.3.2.)