BO1 History of Mathematics
Lecture V
Successes of and difficulties with the calculus: the 18th-century beginnings of ‘rigour’
Part 2: Functions

MT 2021 Week 3

Functions: isoperimeter problem

Classical Problem (Virgil's *Aeneid*): Find the closed curve of given length L that maximises the area enclosed.

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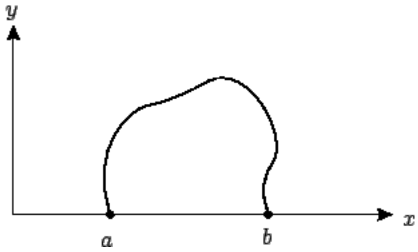

Modern Formulation: Find a function f and corresponding curve $y = f(x)$ between $(a, 0)$ and $(b, 0)$ of given length L (where $L > b - a$) that maximises the area beneath it.

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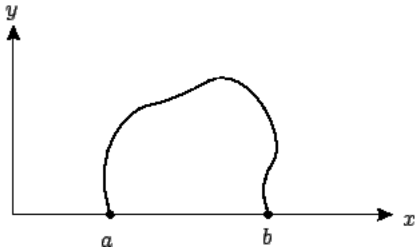

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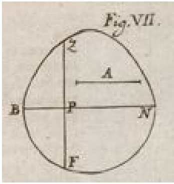
But what is meant by 'function'?

Functions: isoperimeter problem

Isoperimeter problem posed by Jacob Bernoulli to Johann Bernoulli, May 1697, verbally and geometrically (ratio and proportion)



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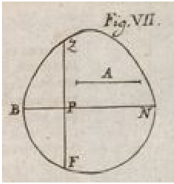

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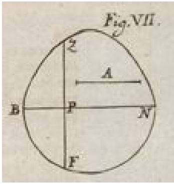

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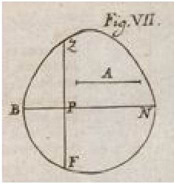

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In 1718, gave the following definition:

> *Here one calls a function of a variable magnitude, a quantity composed in any manner possible from this variable magnitude and constants.*

(See *Mathematics emerging*, §9.1.1.)

Functions: the wave equation

Another success of calculus: the wave equation

$$\frac{\partial^2 y}{\partial s^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

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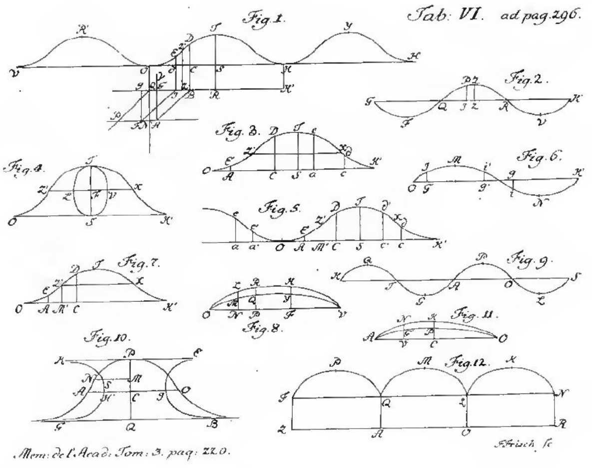
Solved by d'Alembert (1747) and Euler (1748) with solutions of the form

$$y = \Psi(s + ct) - \Phi(s - ct).$$

Functions: the wave equation



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- differentiable?

Functions: the wave equation

But which 'functions' are admissible as solutions?

Must they be

- continuous?
- differentiable?
- ... whatever these mean ...

What is a function?

Euler's definition of a function (1748):

> *A function of a variable quantity is an analytic expression composed in any way from that variable quantity and from numbers or constant quantities.*
>
> ...
>
> *Functions are divided into algebraic and transcendental; the former are those composed by algebraic operations alone, but the latter are those in which transcendental operations are involved.*

L. Euler: *Introductio in analysin infinitorum* (1748) [*Introduction to the analysis of the infinite*], available in translation, Springer-Verlag, 1988.

What is a function?

Euler's new definition of a function (1755):

> *Moreover, the quantities that depend in this way on others, so that the latter having changed, they themselves also undergo change, are usually called functions; which name opens up most generally all the ways in which one quantity may be determined from others involved with it.*

L. Euler: *Institutiones calculi differentialis* [*Foundations of differential calculus*] (1755)

What is a function?

In fact, this question took a long time to settle.

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[For a list of different definitions of functions, ranging from 1718 to 1939, see: Dieter Rüthing, Some definitions of the concept of function from Joh. Bernoulli to N. Bourbaki, *The Mathematical Intelligencer* **6**(4) (1984) 72–77]