BO1 History of Mathematics
Lecture VI
Infinite series
Part 2: The 17th century

MT 2021 Week 3

Infinite series 1600–1900: an overview

Lecture VI:

- mid–late 17th century: many discoveries
- early 18th century: much progress
- later 18th century: doubts and questions

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- mid–late 17th century: many discoveries
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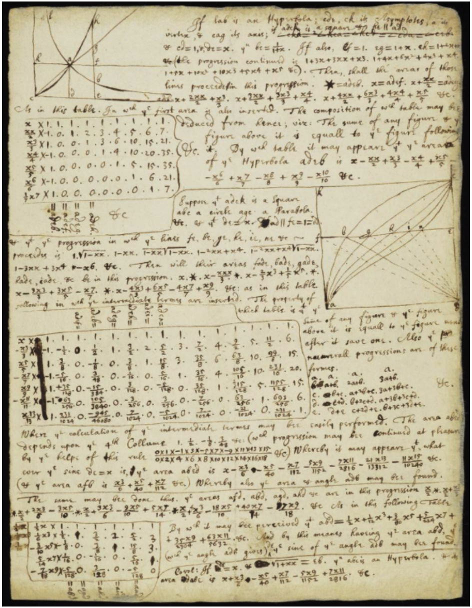
Lecture VII:

- early 19th century: Fourier series
- early 19th century: convergence better understood

Newton and the general binomial theorem

CUL Add. MS 3958.3, f. 72

(See lecture IV)



Recall: Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int(1+x)^n dx$. (NB. Newton did not use the notation $\int(1+x)^n dx$.)

Newton and the general binomial theorem

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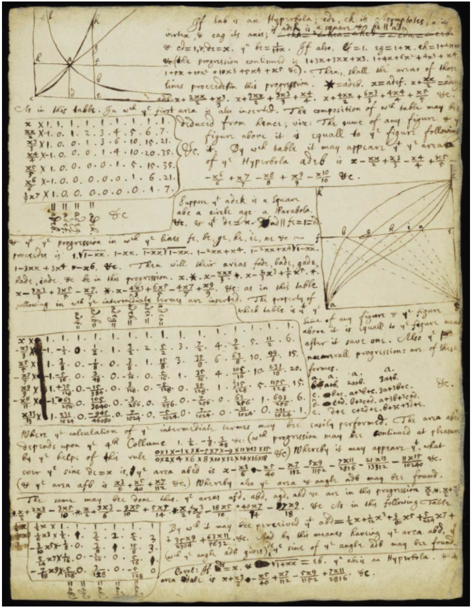
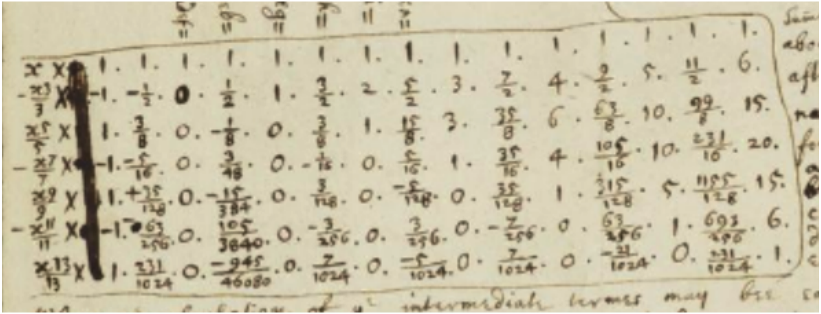


Newton's method of interpolation



x X	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
$-\frac{x^3}{3}$ X	-1.	$-\frac{1}{2}$.	0.	$\frac{1}{2}$.	1.	$\frac{3}{2}$.	2.	$\frac{5}{2}$.	3.	$\frac{7}{2}$.	4.	$\frac{9}{2}$.	5.	$\frac{11}{2}$.	6.
$\frac{x^5}{5}$ X	1.	$\frac{3}{8}$.	0.	$-\frac{1}{8}$.	0.	$\frac{3}{8}$.	1.	$\frac{15}{8}$.	3.	$\frac{35}{8}$.	6.	$\frac{63}{8}$.	10.	$\frac{99}{8}$.	15.
$-\frac{x^7}{7}$ X	-1.	$-\frac{5}{16}$.	0.	$\frac{3}{48}$.	0.	$-\frac{1}{16}$.	0.	$\frac{5}{16}$.	1.	$\frac{35}{16}$.	4.	$\frac{105}{16}$.	10.	$\frac{231}{16}$.	20.
$\frac{x^9}{9}$ X	1.	$+\frac{35}{128}$.	0.	$-\frac{15}{384}$.	0.	$\frac{3}{128}$.	0.	$-\frac{5}{128}$.	0.	$\frac{35}{128}$.	1.	$\frac{315}{128}$.	5.	$\frac{1155}{128}$.	15.
$-\frac{x^{11}}{11}$ X	-1.	$-\frac{63}{256}$.	0.	$\frac{105}{3840}$.	0.	$-\frac{3}{256}$.	0.	$\frac{3}{256}$.	0.	$-\frac{7}{256}$.	0.	$\frac{63}{256}$.	1.	$\frac{693}{256}$.	6.
$\frac{x^{13}}{13}$ X	1.	$\frac{231}{1024}$.	0.	$\frac{-945}{46080}$.	0.	$\frac{7}{1024}$.	0.	$\frac{-5}{1024}$.	0.	$\frac{7}{1024}$.	0.	$\frac{-21}{1024}$.	0.	$\frac{231}{1024}$.	1.

Newton's method of extrapolation

In fact, this method extends easily to any integer n

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Newton's explanation:

> *The property of which table is that the sum of any figure and the figure above it is equal to the figure next after it save one. Also the numerall progressions are of these forms.*

a	a	a	a	
b	$a+b$	$2a+b$	$3a+b$	
c	$b+c$	$a+2b+c$	$3a+3b+c$	$\&c.$
d	$c+d$	$b+2c+d$	$a+3b+3c+d$	
e	$d+e$	$c+2d+e$	$b+3c+3d+e$	

(See: *Mathematics emerging*, §8.1.1.)

Newton's method of interpolation

	$(1-x^2)^{-1}$	$(1-x^2)^{-\frac{1}{2}}$	$(1-x^2)^0$	$(1-x^2)^{\frac{1}{2}}$	$(1-x^2)^1$	$(1-x^2)^{\frac{3}{2}}$	$(1-x^2)^2$	$\cdots$
x	1	1	1	1	1	1	1	$\cdots$
$-\frac{x^3}{3}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\cdots$
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\cdots$
$-\frac{x^7}{7}$	-1	$-\frac{5}{16}$	0	$\frac{3}{48}$	0	$-\frac{1}{16}$	0	$\cdots$
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\pm\frac{x^m}{m}$ and the column labelled $(1-x^2)^n$ is the coefficient of $\pm\frac{x^m}{m}$ in $\int(1-x^2)^n dx$.

(NB: possible slips in the last two rows of the original table)

Newton's method of interpolation

Can fill in some initial values by other methods

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Newton applied the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

to fractional n,

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$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

to fractional n, so that

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{1/2(1/2-1)}{2!} = -\frac{1}{8}$$

and so on

Newton and the general binomial theorem

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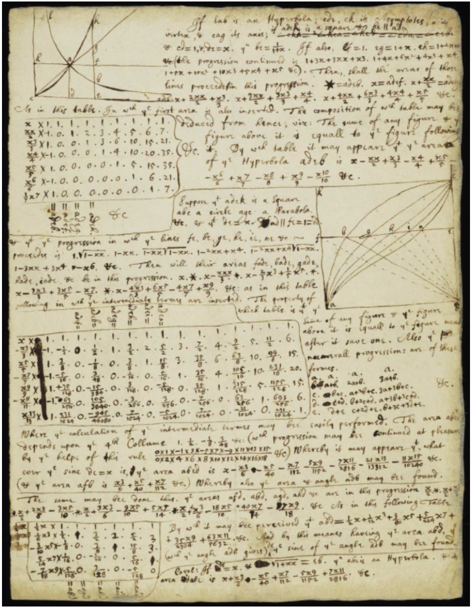
On Newton and the binomial theorem, see https://www.youtube.com/watch?v=xv_PWwdDWDk

One more table



The table at the bottom of the page gives the interpolations for $(1+x)^n$ for half-integer n

Further discoveries by Newton

By further interpolations and integrations (based on strong geometric intuition) Newton found further series for:

- $(1 + x)^{p/q}$

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- $(1+x)^{p/q}$
- log, antilog
- sin, tan, ... (NB: cosine was not yet much in use)
- arcsin, arctan, ...

(See: *Mathematics emerging*, §§8.1.2–8.1.3.)

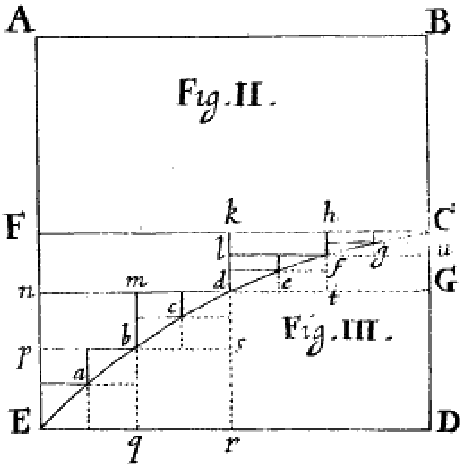
Newton on the move from finite to infinite series

> *And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.*

(*De analysi*, 1669; Derek T. Whiteside, *The mathematical papers of Isaac Newton*, CUP, 1967–1981, vol. II, p. 241)

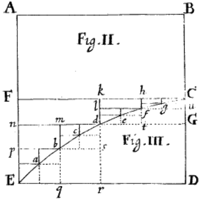
Other 17th-century discoveries (1a)

Brouncker, c. 1655, published 1668: area under the hyperbola given by $\dfrac{1}{1\times 2}+\dfrac{1}{3\times 4}+\dfrac{1}{5\times 6}+\cdots$



$$\left.\begin{array}{l}\text{I say ABCdEA} = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \frac{1}{7\times 8} + \frac{1}{9\times 10} \ \&c. \\ \text{EdCDE} = \frac{1}{2\times 3} + \frac{1}{4\times 5} + \frac{1}{6\times 7} + \frac{1}{8\times 9} + \frac{1}{10\times 11} \ \&c. \\ \text{EdCyE} = \frac{1}{2\times 3\times 4} + \frac{1}{4\times 5\times 6} + \frac{1}{6\times 7\times 8} + \frac{1}{8\times 9\times 10} \ \&c.\end{array}\right\} \textit{in infinitum.}$$

(647)



And that therefore in the first series half the first term is greater than the sum of the two next, and half this sum of the second and third greater than the sum of the four next, and half the sum of those four greater than the sum of the next eight, *&c. in infinitum.* For $\frac{1}{2}\, dD = br + bn$; but $bn > fG$, therefore $\frac{1}{2}\, dD > br + fG$, *&c.* And in the *second* series half the first term is less then the sum of the two next, and half this sum less then the sum of the four next, *&c. in infinitum.*

That the first *series* are the *even* terms, *viz.* the 2d, 4th, 6th, 8th, 10th, *&c.* and the *second*, the *odd*, *viz.* the 1st, 3d, 5th, 7th, 9th, *&c.* of the following series, *viz.* $\frac{1}{1\times 2}$. $\frac{1}{2\times 3}$. $\frac{1}{3\times 4}$. $\frac{1}{4\times 5}$. $\frac{1}{5\times 6}$. $\frac{1}{6\times 7}$. *&c. in infinitum* = 1. Whereof *a* being put for the number of terms taken at pleasure, $\frac{1}{aa + a}$ is the last, $\frac{a}{a+1}$ is the sum of all those terms from the beginning, and $\frac{1}{a+1}$ the sum of the rest to the end.

That $\frac{1}{4}$ of the first terme in the *third* series is less than the sum of the two next, and a quarter of this sum, less than the sum of the four next, and one fourth of this last sum less than the next eight, I thus demonstrate.

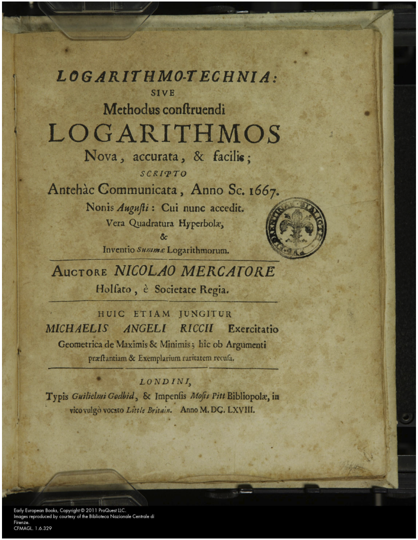
Let a = the 3d or last number of any term of the first Column, *viz.* of Divisors,

Other 17th-century discoveries (2)

Mercator's series (1668), found by long division:

$$\frac{1}{1+a} = 1 - a + aa - a^3 + a^4 \ (\&c.)$$

Gives rise to series for log


LOGARITHMO-TECHNIA:
SIVE
Methodus conſtruendi
LOGARITHMOS
Nova, accurata, & facilis;
SCRIPTO
Antehàc Communicata, Anno Sc. 1667.
Nonis *Auguſti*: Cui nunc accedit.
Vera Quadratura Hyperbolæ,
&
Inventio *Summæ* Logarithmorum.
AUCTORE *NICOLAO MERCATORE*
Holſato, è Societate Regia.
HUIC ETIAM JUNGITUR
MICHAELIS ANGELI RICCII Exercitatio
Geometrica de Maximis & Minimis; hîc ob Argumenti
præſtantiam & Exemplarium raritatem recuſa.
LONDINI,
Typis *Guilielmi Godbid*, & Impenſis *Moſis Pitt* Bibliopolæ, in
vico vulgò vocato *Little Britain*. Anno M.DC.LXVIII.

Other 17th-century discoveries (3)

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- series for tan, sec, and others, including

$$\theta = \tan\theta - \frac{1}{2}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

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Gregory to Collins, 23rd November 1670:

> *I suppose these series I send here enclosed, may have some affinity with those inventions you advertise me that Mr. Newton had discovered.*

(On Gregory's work, see: *Mathematics emerging*, §8.1.4.)

Other 17th-century discoveries (4)

Gottfried Wilhelm Leibniz (1675):

The area of a circle with unit diameter is given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$$

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The error in the sum is successively less than $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, etc.

> *Therefore the series as a whole contains all approximations at once, or values greater than correct and less than correct: for according to how far it is understood to be continued, the error will be smaller than a given fraction, and therefore also less than any given quantity. Therefore the series as a whole expresses the exact value.*

(See: *Mathematics emerging*, §8.3.)

Series in the 17th century: 'convergence'

John Wallis (1656), *Arithmetica infinitorum*:

$$\square = \frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \cdots}$$

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and so on)

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Leibniz (1675): 'alternating' series

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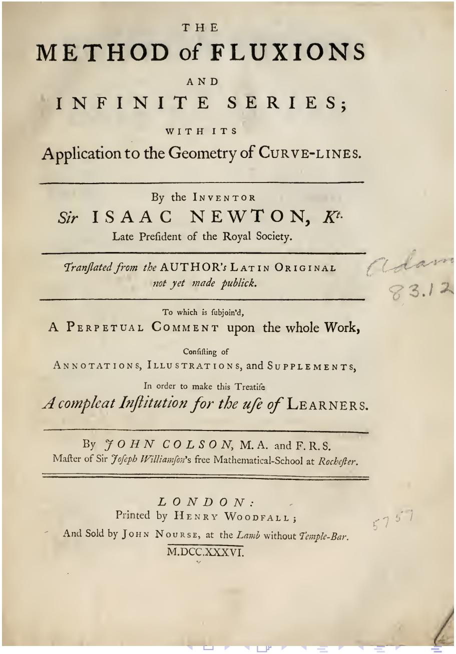
Power series rank with calculus as a major advance of the 17th century

Calculus and series combined

Newton's treatise of 1671, published 1736



THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, *Kt.*

Late Prefident of the Royal Society.

Tranflated from the AUTHOR'*s* LATIN ORIGINAL *not yet made publick.*

To which is fubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Confifting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatife

A compleat Inftitution for the ufe of LEARNERS.

By *JOHN COLSON*, M. A. and F. R. S.

Mafter of Sir *Jofeph Williamfon*'s free Mathematical-School at *Rochefter*.

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M.DCC.XXXVI.