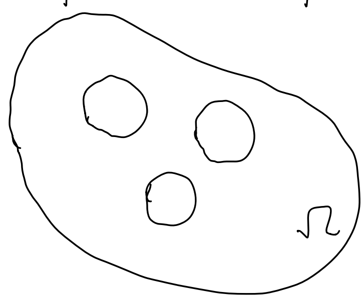
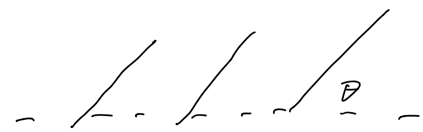
2.6.2 Uniformasation of multiply connected domain.

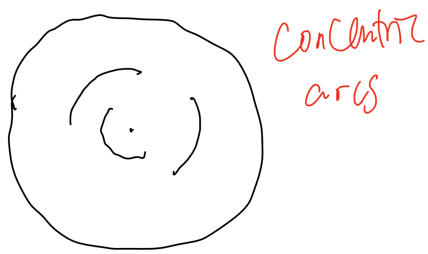


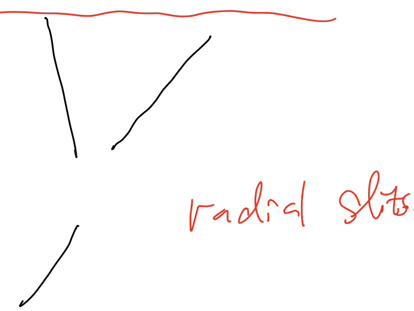


Let Ω be a multiply connected domain. z_0 be some pt in Ω, θ be an angle in $[0, 2\pi)$. Then there is a unique univalent map $f_{z_0,\theta} : \Omega \to SD$ where θ is the angle of the slits with the real axis, z_0 is mapped to ∞, and the Laurent expansion of the map at z_0 is of the form

$$f_{z_0,\theta}(z) = \frac{1}{z-z_0} + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

For the mapping onto the parallel slit domain, the class of admissible funtions is the class of all univalent functions in Ω which takes the above form (for the Laurent expansion at z_0) the function $f_{z_0,\theta}$ has the maximal value of $\mathrm{Re}(e^{-2i\theta} a_1)$





2.7 Boundary correspondence.

Prop 2.7.1 Let f be a univalent map from $\Omega \xrightarrow{\text{onto}} \Omega'$ and let $z_n \in \Omega$ be a seq which tends to the bdry of Ω

then $f(z_n)$ tends to the bdry of Ω'.

(all accumulation pts are on the bdry of Ω;
for any $z \in \Omega$, there exist an $\varepsilon > 0$, and $n_0 \in \mathbb{N}$,
such that $|z_n - z| \geq \varepsilon$ if $n > n_0$) z_n Baway from any $z \in \Omega$.
" $z(t)$ $t > t_0$ "

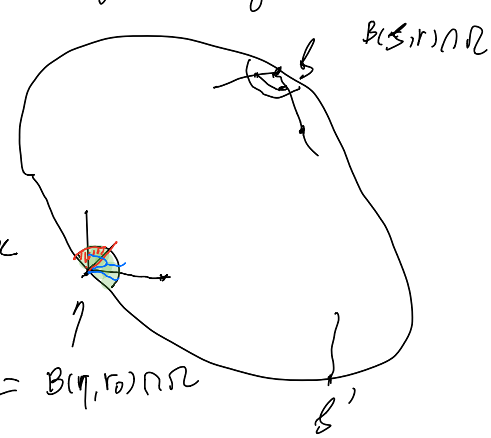
proof: let K be a compact set of Ω'. Then $f^{-1}(K)$ is a compact set in Ω. and $\exists\, n_0$ such that $z_n \notin f^{-1}(K)$ if $n > n_0$. Then $f(z_n)$ is not in K. $\Rightarrow$ $f(z_n)$ tends to $\partial\Omega'$.

$f^{-1}(K)$ compact $\exists\, n_0$, $z_n \notin f^{-1}(K)$ for all $n > n_0$. $f(z_n) \notin K$,

2.7.1 Accessible pts.

Def 2.7.2. An accessible bdry pt ζ of domain Ω is an equivalence class of continuous curves $\gamma: [0,1] \to \Omega$, which join a given pt $\zeta \in \partial\Omega$ with an arbitrary interior point. We assume γ lies completely inside Ω except at $\gamma(1) = \zeta$.

Two curves are equivalent if for any small nbhd of ζ, part of the curves that are inside $U \cap \Omega$ could be joined by a continuous curve there in.



RMK (1) Accessible pts corresponding to different bdry pts are different. but the same bdry pt could have several different accessible pts.

(2) For two accessible pts there are nbhds seperating them

$= B(\eta, r_0) \cap \Omega$

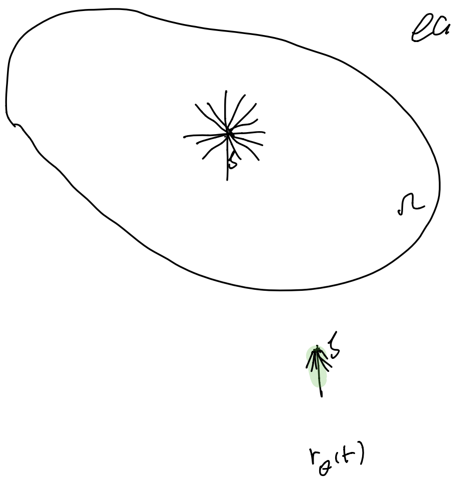
$U(\eta, r_0)$

but if there are infinitely many accessible pts corresponding to the same bdry pts, then there could be no single "γ_0" which seperate a particular accessible pt from all others.

ex: $\zeta = 0$ $\quad \theta = 2\pi k/2^n \quad [0, e^{i\theta}/2^n]$

each irrational (mod 2π) angle θ

$r_\theta(t) = (1-t)e^{i\theta}$ accessible pts.



Thm 2.7.3 Let Ω be a simply connected bdd domain in the complex plane. and f be a univalent map $\Omega \to \mathbb{D}$. Then for every accessible pt ζ, the map f can be continuously extended to ζ and $|f(\zeta)| = 1$. Moreover for distinct accessible pts, the images are different. ! !

Lemma 2.7.4 (Koebe) Let z_n, z_n' be two seqs in $\mathbb{D}$ converging to distinct pts ζ, ζ' on the unit circle.

Let γ_n be the Jordan arc connecting z_n, and z_n' inside $\mathbb{D}$, but outside some fixed nbhd of 0.

Assume f analytic, bdd in $\mathbb{D}$ and f converges uniformly to 0 on γ_n. i.e. $\varepsilon_n = \sup_{\gamma_n} |f| \to 0$

$\Rightarrow f \equiv 0$ on $\mathbb{D}$.

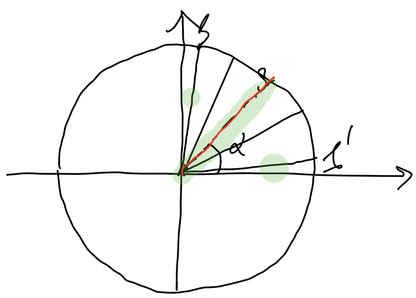
(Jordan arc: is the image of an injective, continuous map of a closed bdd interval $[a,b]$ to the plane).

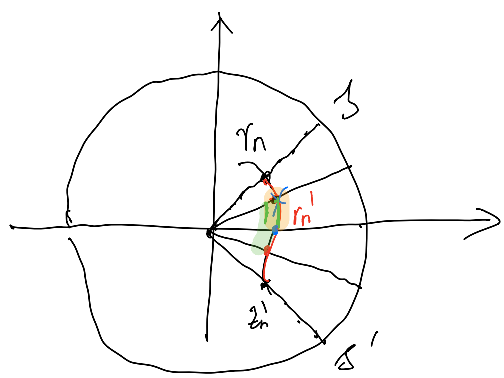
proof: Suppose f is not identically 0. Then either $f(0) \neq 0$.

or. $z_0 = 0$ is a zero of order $n < \infty$ for f.

(for the 2nd case, we replace f in the following proof by f/z^n).

For sufficiently large $m \in \mathbb{N}$. $\exists$ a sector S of angle $2\pi/m$ such that the radii connecting ζ, ζ' are both outside of S.





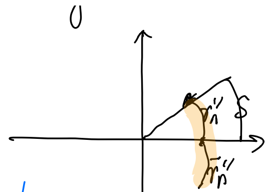
by performing $f(e^{i\alpha} z)$ instead of $f(z)$

we can assume the line bisecting S lies in the real line.

take γ_n' of γ_n which is a simple curve that crosses S with endpts lying on two different sides of S, and there is no other pt in γ_n' lying on the bdry of S.

Let γ_n'' be the part of γ_n' connecting one of the endpt of the side of sector S with the first intersection of γ_n' with the real line.

By ~~reflection principle~~. $f(\bar{z})$ is analytic in $\mathbb{D}$ and bdd by ε_n on $\bar{\gamma}_n''$



$|\bar f(\bar z)| \le \varepsilon_n$ on $\bar\gamma_n''$ because reflection principle

$\phi(z) = f(z)\bar f(\bar z)$ analytic, bdd on the union of γ_n'' and $\bar\gamma_n''$

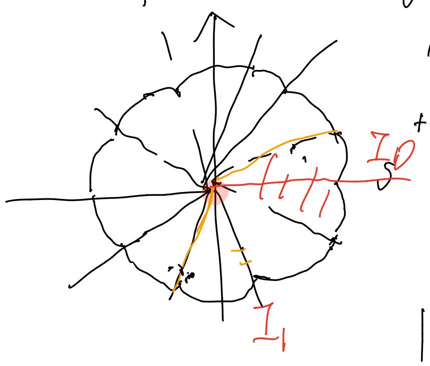
real on real line and $\overline{\phi(\bar z)} = \phi(z)$

$|\phi(z)| \le M\varepsilon_n$ where $M = \sup_{\mathbb{D}} |f|$

Let F be the product to rotation of ϕ by $2\pi/m$.

(*) $F(z) = \phi(z)\,\phi(e^{\frac{2\pi i}{m}} z) \cdots \phi(e^{\frac{2\pi i (m-1)}{m}} z)$

The function is analytic in $\mathbb{D}$ and bdd by $\varepsilon_n M^{2m-1}$



in the region bdd by the closed curve. (formed by rotating $\gamma_n'' \cup \bar\gamma_n''$) $\Uparrow$

by maximum principle.

$|f(0)|^{2m} = |F(0)| \le \varepsilon_n M^{2m-1}$
$\to 0$
as $n \to \infty$ $\Rightarrow$ $f(0) = 0$

contradiction.

Fix (*).

Let $\tilde\phi(z) = \phi_i(z)$ in I_i

$\phi_0(z) = \phi(z)$ in I_0

$\phi_1(z) = \phi(z e^{i\frac{2\pi}{m}})$ in I_1

$\phi_k(z) = \phi(z e^{\frac{2k\pi i}{m}})$ in I_k

$F(z) = \tilde\phi(z)$

$|f(0)|^2 = |F(0)| \le \varepsilon_n M \to 0$ as $n \to \infty$