C4.3 Functional Analytic Methods for PDEs
Lecture 7

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In the last 2 lectures

- Definition of Sobolev spaces
- Extension theorems for Sobolev functions.

This lecture

- Trace (boundary value) of Sobolev functions.
- Gagliardo-Nirenberg-Sobolev's inequality

More on extension

Theorem (Stein's extension theorem)

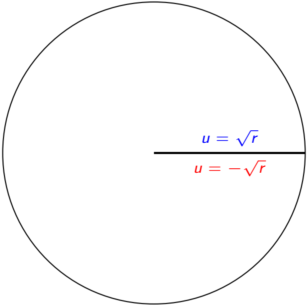
Assume that Ω *is a bounded Lipschitz domain. Then there exists a linear operator* E *sending functions defined a.e. in* Ω *to functions defined a.e. in* $\mathbb{R}^n$ *such that for every* $k \geq 0$, $1 \leq p < \infty$ *and* $u \in W^{k,p}(\Omega)$ *it hold that* $Eu = u$ *a.e. in* Ω *and*

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,p,\Omega}\|u\|_{W^{k,p}(\Omega)}$$

The operator E is called a total extension for Ω.

More on extension

- There exists domain Ω for which there is no bounded linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ such that $Eu = u$ a.e. in Ω.



$\Omega = \{x^2 + y^2 < 1\} \setminus \{(x,0) | x \geq 0\}$

$\bar{\Omega} = \{x^2 + y^2 \leq 1\}$

$D = \{x^2 + y^2 < 1\}$

We knew that the function $u(x,y) = \sqrt{r} \cos \frac{\theta}{2}$ satisfies

- ⋆ $u \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$.
- ⋆ $u \notin W^{1,2}(D)$.

So no extension of u belongs to $W^{1,2}(\mathbb{R}^2)$.

Values of Sobolev functions on the boundary

- As prompted at the beginning of the course, in our later applications in the analysis of PDEs, solutions will live in a Sobolev space.
- When discussing PDEs on a domain, one needs to specify boundary conditions.
- A complication arises:
  - ⋆ On one hand, Sobolev 'functions' are equivalent classes of functions which are equal almost everywhere. Thus one can redefine the value of a Sobolev function on set of measure zero at will without changing the equivalent class it represents.
  - ⋆ On the other hand, the boundary of a domain usually has measure zero. So the boundary value of a Sobolev function cannot simply be defined by restricting as is the case for continuous functions.

Values of Sobolev functions on the boundary

Remark

Suppose $1 \le p < \infty$, Ω *is a bounded smooth domain and let* $(X, \|\cdot\|)$ *be a normed vector space which contains* $C(\partial\Omega)$. *There is NO <u>bounded</u> linear operator* $T : L^p(\Omega) \to X$ *such that* $Tu = u|_{\partial\Omega}$ *for all* $u \in C(\bar{\Omega})$.

Proof

- Suppose by contradiction that such T exists. Consider $f_m \in C(\bar{\Omega})$ defined by

$$f_m(x) = \begin{cases} m\, dist(x, \partial\Omega) & \text{if } dist(x, \partial\Omega) < 1/m, \\ 1 & \text{if } dist(x, \partial\Omega) \ge 1/m. \end{cases}$$

- Then $\|f_m - 1\|_{L^p(\Omega)}^p \le |\{dist(x, \partial\Omega) < 1/m\}| \le \frac{C}{m}$ and so $f_m \to 1$ in $L^p(\Omega)$.
- Now as $Tf_m = 0 \not\to 1 = T1$ in X, T cannot be bounded.

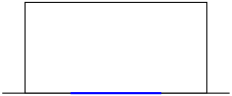
Values of Sobolev functions on the boundary

Theorem

Suppose $1 \le p < \infty$, and that Ω is a bounded Lipschitz domain. Then there exists a <u>*bounded*</u> *linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called the trace operator, such that $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$.*

We will only proof a weaker statement in a simpler situation:

$\Omega = \{x = (x', x_n) : |x'| < 2, 0 < x_n < 2\}$



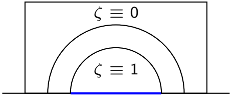
$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$
$\Gamma = \{x = (x', 0) : |x'| < 1\}$

We would like to define the trace operator relative to Γ: There exists a bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ such that

$$T_\Gamma u = u|_\Gamma \text{ for all } u \in C^1(\bar{\Omega}).$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2, \; 0 < x_n < 2\}$



$\hat\Gamma = \{x = (x', 0) : |x'| < 2\}$

$\Gamma = \{x = (x', 0) : |x'| < 1\}$

$0 \leq \zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1

- We first prove the key estimate

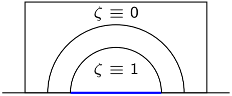
$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar\Omega). \qquad (*)$$

  - ⋆ We have

$$\int_\Gamma |u|^p \, dx' \leq \int_{\hat\Gamma} \zeta |u|^p \, dx' = -\int_{\hat\Gamma} \Big[\int_0^2 \partial_{x_n}(\zeta |u|^p) \, dx_n \Big] \, dx'$$
$$= -\int_\Omega \partial_{x_n}(\zeta |u|^p) \, dx \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx.$$

Values of Sobolev functions on the boundary

$\Omega = \{x = (x', x_n) : |x'| < 2,$
$0 < x_n < 2\}$



$\hat{\Gamma} = \{x = (x', 0) : |x'| < 2\}$
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$\zeta \in C_c^\infty(B_{3/2})$ such that $\zeta \equiv 1$ in B_1.

- We first prove the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

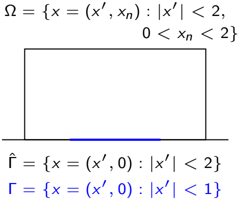
  ⋆ We have $\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du||u|^{p-1}] \, dx$.

  ⋆ Using the inequality $|a||b|^{p-1} \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^p$, we obtain

$$\int_\Gamma |u|^p \, dx' \leq C_{p,\zeta} \int_\Omega [|u|^p + |Du|^p] \, dx$$

  This proves (*).

Values of Sobolev functions on the boundary



- We have proved the key estimate

$$\|u\|_{L^p(\Gamma)} \leq C_p \|u\|_{W^{1,p}(\Omega)} \text{ for all } u \in C^1(\bar{\Omega}). \qquad (*)$$

- It follows that the map $u \mapsto u|_\Gamma =: Au$ is a bounded linear operator from $(C^1(\bar{\Omega}), \|\cdot\|_{W^{1,p}})$ into $L^p(\Gamma)$.
- As Ω is Lipschitz, $C^\infty(\bar{\Omega})$ and hence $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists a unique bounded linear operator $T_\Gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ which extends A, i.e. $T_\Gamma u = u|_\Gamma$ for all $u \in C^1(\bar{\Omega})$.

IBP formula revisited

Proposition (Integration by parts)

Suppose that $1 \le p < \infty$, Ω is a bounded Lipschitz domain, n be the outward unit normal to $\partial\Omega$, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then

$$\int_\Omega \partial_i u\, v\, dx = \int_{\partial\Omega} Tu\, v\, n_i\, dS - \int_\Omega u\, \partial_i v\, dx \text{ for all } v \in C^1(\bar{\Omega}).$$

Proof

- We knew that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. Thus there exists $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{1,p}$.
- Fix some $v \in C^1(\bar{\Omega})$. We have

$$\int_\Omega \partial_i u_m\, v\, dx = \int_{\partial\Omega} u_m\, v\, n_i\, dS - \int_\Omega u_m\, \partial_i v\, dx.$$

IBP formula revisited

Proof

- $\int_\Omega \partial_i u_m \, v \, dx = \int_{\partial\Omega} u_m \, v \, n_i \, dS - \int_\Omega u_m \, \partial_i v \, dx$.
- Note that $\partial_i u_m \to \partial_i u$, $u_m \to u$ in $L^p(\Omega)$ and $u_m|_{\partial\Omega} = Tu_m \to Tu$ in $L^p(\partial\Omega)$. We can thus argue using Hölder's inequality to send $m \to \infty$ to obtain

$$\int_\Omega \partial_i u \, v \, dx = \int_{\partial\Omega} Tu \, v \, n_i \, dS - \int_\Omega u \, \partial_i v \, dx$$

as wanted.

Functions of zero trace

Theorem (Trace-zero functions in $W^{1,p}$)

Suppose that $1 \leq p < \infty$, Ω is a bounded Lipschitz domain, $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is the trace operator, and $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$.

Proof

- ($\Rightarrow$) Suppose $u \in W_0^{1,p}(\Omega)$. By definition, there exists $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{1,p}$. Clearly $Tu_m = 0$ and so by continuity, $Tu = 0$.
- ($\Leftarrow$) We will only consider the case Ω is the unit ball B. This proof can be generalised fairly quickly to star-shaped domains. The proof for Lipschitz domains is more challenging.

Functions of zero trace

Proof

- ($\Leftarrow$) Suppose that $u \in W^{1,p}(B)$ and $Tu = 0$. We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Let $\bar{u}$ be the extension by zero of u to $\mathbb{R}^n$.
  - ⋆ As $Tu = 0$, we have by the IBP formula that

    $$\int_B \partial_i u\, v\, dx = -\int_B u\, \partial_i v\, dx \text{ for all } v \in C^1(\bar{B}).$$

    It follows that

    $$\int_B \partial_i u\, v\, dx = -\int_B \bar{u}\, \partial_i v\, dx \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

    By definition of weak derivatives, this means

    $$\partial_i \bar{u} = \begin{cases} \partial_i u & \text{in } B \\ 0 & \text{elsewhere} \end{cases} \quad \text{in the weak sense.}$$

    So $\bar{u} \in W^{1,p}(\mathbb{R}^n)$.

Functions of zero trace

Proof

- ($\Leftarrow$) We would like to construct a sequence $u_m \in C_c^\infty(B)$ such that $u_m \to u$ in $W^{1,p}(B)$.
  - ⋆ Let $\bar{u}_\lambda(x) = \bar{u}(\lambda x)$. Observe that $Supp(\bar{u}_\lambda) \subset B_{1/\lambda}$.
  - ⋆ In Sheet 1, you showed that $\bar{u}_\lambda \to \bar{u}$ in L^p as $\lambda \to 1$.
    Noting also that $\partial_i \bar{u}_\lambda(x) = \lambda \partial_i u(\lambda x)$, we also have that $\partial_i \bar{u}_\lambda \to \partial_i \bar{u}$ in L^p as $\lambda \to 1$.
    Hence $\bar{u}_\lambda \to \bar{u}$ in $W^{1,p}$ as $\lambda \to 1$.
  - ⋆ Fix $\lambda_m > 1$ such that $\|\bar{u}_{\lambda_m} - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - ⋆ Let (ϱ_ε) be a family of mollifiers: $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$ with $\varrho \in C_c^\infty(B)$, $\int_{\mathbb{R}^n} \varrho = 1$. Then $\bar{u}_{\lambda_m} * \varrho_\varepsilon \to \bar{u}_{\lambda_m}$ in $W^{1,p}$ as $\varepsilon \to 0$. Also, $Supp(\bar{u}_{\lambda_m} * \varrho_\varepsilon) \subset B_{\lambda_m^{-1}+\varepsilon}$. Thus, we can select ε_m sufficiently small such that $u_m := \bar{u}_{\lambda_m} * \varrho_{\varepsilon_m} \in C_c^\infty(B)$ and $\|u_m - \bar{u}_{\lambda_m}\|_{W^{1,p}(\mathbb{R}^n)} \leq 1/m$.
  - ⋆ Now $\|u_m - u\|_{W^{1,p}(B)} \leq 2/m$ and so we are done.

Embeddings

Let X_1 and X_2 be two Banach spaces.

- We say X_1 is embedded in X_2 if $X_1 \subset X_2$.
- We say X_1 is continuously embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a bounded linear operator, i.e. there exists a constant C such that $\|x\|_{X_2} \leq C\|x\|_{X_1}$. We write $X_1 \hookrightarrow X_2$.
- We say X_1 is compactly embedded in X_2 if X_1 is embedded in X_2 and the identity map $I : X_1 \to X_2$ is a compact bounded linear operator. This means that I is continuous and every bounded sequence $(x_n) \subset X_1$ has a subsequence which is convergent with respect to the norm on X_2.

Our interest: The possibility of embedding $W^{k,p}$ in L^q or C^0.

Gagliardo-Nirenberg-Sobolev's inequality

Theorem (Gagliardo-Nirenberg-Sobolev's inequality)

Assume $1 \leq p < n$ and let $p^* = \frac{np}{n-p}$. Then there exists a constant $C_{n,p}$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in W^{1,p}(\mathbb{R}^n).$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

The number $p^* = \frac{np}{n-p}$ is called the Sobolev conjugate of p. It satisfies $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.
The case $p = 1$ is referred to as Gagliardo-Nirenberg's inequality.

GNS's inequality – Why $p < n$ and why p^*?

Question

For what p and q does it hold

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n)? \qquad (*)$$

This will be answered by a scaling argument:

- Fix a non-zero function $u \in C_c^\infty(\mathbb{R}^n)$. Define $u_\lambda(x) = u(\lambda x)$. Then $u_\lambda \in C_c^\infty(\mathbb{R}^n)$ and so

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute

$$\|u_\lambda\|_{L^q}^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \, dy = \lambda^{-n} \|u\|_{L^q}^q.$$

GNS's inequality – Why $p < n$ and why p^*?

- $u_\lambda(x) = u(\lambda x)$ and

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,q} \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}. \qquad (**)$$

- We compute $\|u_\lambda\|_{L^q} = \lambda^{-n/q} \|u\|_{L^q}$.
- Next,

$$\|\nabla u_\lambda\|_{L^p}^p = \int_{\mathbb{R}^n} |\lambda \nabla u(\lambda x)|^p \, dx$$
$$= \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy = \lambda^{p-n} \|\nabla u\|_{L^p}^p.$$

That is $\|\nabla u_\lambda\|_{L^p} = \lambda^{1-n/p} \|\nabla u\|_{L^p}$.

GNS's inequality – Why $p < n$ and why p^*?

- Putting in (**), we get

$$\lambda^{-n/q}\|u\|_{L^q} \leq C_{n,p,q}\lambda^{1-n/p}\|\nabla u\|_{L^p}.$$

Rearranging, we have

$$\lambda^{-1+\frac{n}{p}-\frac{n}{q}} \leq \frac{C_{n,p,q}\|\nabla u\|_{L^p}}{\|u\|_{L^q}}.$$

- Since the last inequality is valid for all λ, we must have that $-1 + \frac{n}{p} - \frac{n}{q} = 0$, i.e. $q = \frac{np}{n-p} = p^*$. As $q > 0$, we must also have $p \leq n$.
- We conclude that a <u>necessary</u> condition in order for the inequality (*) to hold is that $p \leq n$ and $q = p^*$.