BO1 History of Mathematics
Lecture IX
The 19th-century beginnings of ‘modern algebra’

MT 2021 Week 5

Summary

Part 1

- Lagrange's ideas (1770/71)
- Cauchy and substitutions (1815)
- 'Classical age' of theory of equations 'ends' (1799–1826)

Part 2

- The invention of groups by Galois and Cauchy

Part 3

- 'Symbolical algebra'
- Groups, rings, and fields: the emergence of 'modern algebra' (1854–1900)

Part 1: Resolvents and Permutations

‘Modern’ or ‘abstract’ algebra

19th century: emergence of mathematics whose subject-matter is no longer space or number:

- permutations
- abstract structures (groups, rings, fields, ...)
- linear algebra [see Lecture X]

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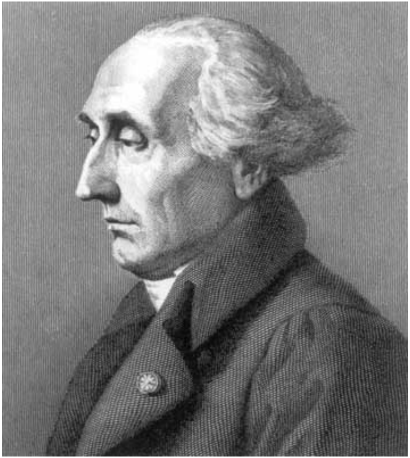
J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/71):



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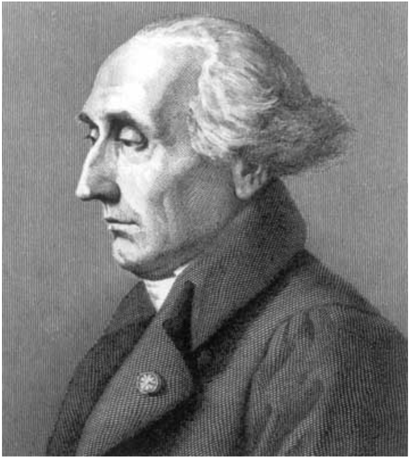
Asserted that there had been little advance in equation-solving since Cardano, but that there was little left to do

Examined all known methods of solving cubics and quartics

Found that in every case the solutions of the 'reduced' (or 'resolvent') equation are 'functions' of the roots of the equation to be solved



Resolvents for cubics

For a cubic with roots x_1, x_2, x_3 there is a reduced equation whose roots are values of

$$y = \frac{1}{3}\left(x_1 + \alpha^2 x_2 + \alpha x_3\right)$$

where $\alpha^3 = 1$, $\alpha \neq 1$.

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Lagrange identified this idea as a feature common to the methods for solving cubics presented by Cardano, Tschirnhaus, Bézout, and Euler

Resolvents for quartics

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Note 2: This theorem mutated several times, finally morphing into Lagrange's Theorem in group theory.

(See *Mathematics emerging*, §12.3.1, and also: Richard L. Roth, A History of Lagrange's Theorem on Groups, *Mathematics Magazine* **74**(2) (2001), pp. 99–108)

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Is there any hope of reducing it further?

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A long, confused and confusing account, which seems to have persuaded no-one except Italian pupils and colleagues?

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Proved his conjecture for $n = 6$

(See *Mathematics emerging*, §13.1.1.)

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The end of 'classical algebra' (?)

Part 2: Groups

Évariste Galois (1811–1832)

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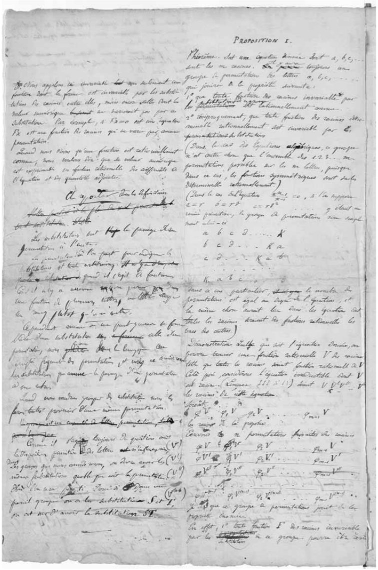
- showed how to associate a group to a polynomial (its Galois group)
- discovered a necessary and sufficient condition for solubility of an equation by radicals expressed in terms of the structure of its group
- as an application, gave a necessary and sufficient condition for solubility of an irreducible equation of prime degree by radicals

Galois and his groups (3)



Premier mémoire, dossier 1, folio 3 verso

Proposition I relates a given polynomial to a group of permutations (its Galois group)

Eleventh-hour marginal additions provide further explanation

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> *Substitutions are the passage from one permutation to another.*
>
> *The permutation from which one starts in order to indicate substitutions is completely arbitrary, ...*
>
> *... one must have the same substitutions, whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST.*

Évariste Galois, 29th May 1832, published 1846

(See *Mathematics emerging*, §13.1.2.)

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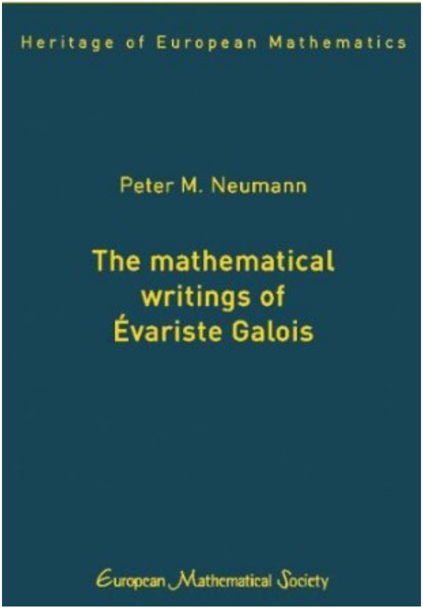
25 Oct 2011:	English/French bilingual edition by Peter Neumann

Galois in English (2011)



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(Peter M. Neumann, ‘On the date of Cauchy’s contributions to the founding of the theory of groups’, *Bull. Austral. Math. Soc.* **40** (1989), 293–302.)

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His purpose: for any function $f(x_1, x_2, \ldots, x_n)$ the substitutions that leave it unchanged (yielding 'valeurs égales') form such a system. The number of values of the function ('valeurs différentes') is the index of this system — that is $\frac{n!}{N}$, where N is the number of its members.

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Hence — a proof of his 1815 conjecture and more.

The Paris Grand Prix of 1860

Académie des Sciences, Paris, *Grand Prix de Mathématiques*, 1860: subject announced April 1857 (Cauchy on committee, he died a month later):

> *What are the possibilities for the number of values of well defined functions containing a given number of letters, and how can one form the functions for which there exist a given number of values?*

Grand Prix 1860: responses

Three competitors:

- Émile Mathieu;
- Camille Jordan

  (submitted their Paris doctoral dissertations);
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The competition stimulated:

- development of theory of (finite permutation) groups;
- merger of Galois' and Cauchy's independent theories.

Part 3: Emergence of Abstract Algebra

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It first appeared in print in the 1830 *A Treatise of Algebra* by George Peacock (1791–1858)

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But the idea of symbolical algebra had largely faded away by the middle of the century: one *could* work with arbitrary operations in an entirely abstract setting, but why would one want to?

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> *A set of symbols*
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> $$1, \alpha, \beta, \ldots$$
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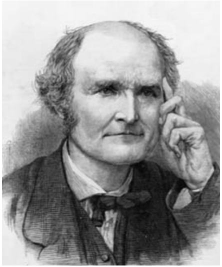
> *all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group.*



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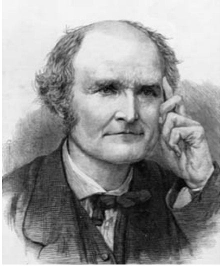

Cayley widely attributed with introducing 'abstract' theory of groups

(See *Mathematics emerging*, §13.1.4.)

Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
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Cayley, 'On the theory of groups' (1878):

> *A group is defined by means of the laws of combinations of its symbols.*

Weber's axioms, 1882

A System G of h elements of any kind, $\Theta_1, \Theta_2, \ldots, \Theta_h$ is called a group of degree h, if it satisfies the following conditions:

I. By some rule, which will be called composition or multiplication, from two elements of the system a new element of the system may be derived. In symbols:

$$\Theta_r \Theta_s = \Theta_t.$$

II. Always:

$$(\Theta_r \Theta_s)\Theta_t = \Theta_r(\Theta_s \Theta_t) = \Theta_r \Theta_s \Theta_t.$$

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Existence of identity and inverses appear as deductions from the axioms — incorporated as axioms by later authors

On axiomatisation of groups

Peter M. Neumann, 'What groups were: a study of the development of the axiomatics of group theory', *Bull. Austral. Math. Soc.* **60** (1999), 285–301.

Christopher D. Hollings, '"Nobody could possibly misunderstand what a group is": a study in early twentieth-century group axiomatics', *Arch. Hist. Exact Sci.* **71**(5) (2017), 409–481.

Rings and ideals

Ernst Kummer (1844):

- concerned with Fermat's last theorem, and quadratic forms
- worked with arithmetic of 'cyclotomic integers' $a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1}$ where θ is primitive n-th root of 1
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Richard Dedekind, 'Sur la théorie des nombres entiers algébriques' (1877) and famous appendices to his editions of Dirichlet's *Lectures on Number Theory*:

- changed Kummer's 'ideal numbers' to 'ideals'
- worked also with rings of numbers [domains] and fields of numbers [Körper]

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Abstract algebra given an early boost (in the USA) via a short-lived obsession with ‘postulate analysis’: the study of systems of axioms for their own sake

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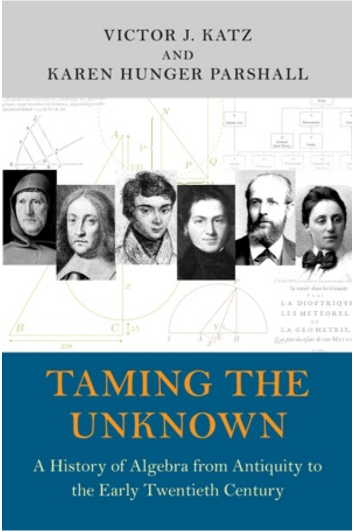
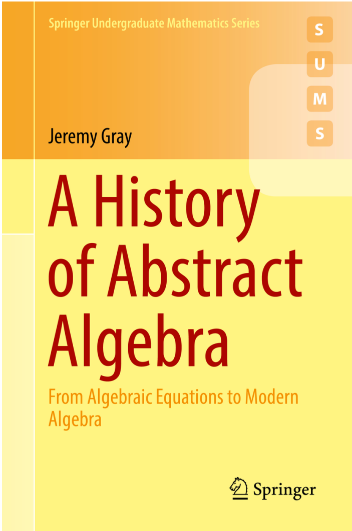
Abstract point of view now dominant, with many different objects studied: groups, fields, rings, integral domains, semigroups, algebras, lattices, semirings, quasigroups, ...

Overviews of the topics of lectures IX and X



(Princeton Univ. Press, 2014)



(Springer, 2018)