C4.3 Functional Analytic Methods for PDEs
Lecture 9

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In the last lecture

- Gagliardo-Nirenberg-Sobolev's inequality, $1 \leq p < n$.
- Trudinger's inequality, $p = n$.
- Morrey's inequality, $n < p \leq \infty$.

This lecture

- Morrey's inequality, $n < p \leq \infty$.
- Friedrichs' inequality.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \le \infty$. *Then every* $u \in W^{1,p}(\mathbb{R}^n)$ *has a* $(1 - \frac{n}{p})$*-Hölder continuous representative. Furthermore there exists a constant* $C_{n,p}$ *such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

An integral mean value inequality

Lemma

Let Ω be a domain in $\mathbb{R}^n$ and suppose $u \in C^1(\Omega)$. Then

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} dy \text{ for all } B_r(x) \subset \Omega.$$

Proof

- It suffices to consider the case $x = 0$. We write $y = s\theta$ where $s \in [0, r)$ and $\theta \in \mathbb{S}^{n-1} \in \mathbb{R}^n$.
- By the fundamental theorem of calculus, we have

$$u(s\theta) - u(0) = \int_0^s \frac{d}{dt} u(t\theta)\, dt = \int_0^s \theta_i \partial_i u(t\theta)\, dt.$$

It follows that $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\, dt$.

An integral mean value inequality

Proof

- $|u(s\theta) - u(0)| \leq \int_0^s |\nabla u(t\theta)|\, dt.$
- Integrating over θ and using Tonelli's theorem, we get

$$\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta \leq \int_0^s \int_{\partial B_1(0)} |\nabla u(t\theta)|\, d\theta\, dt$$
$$= \int_0^s \int_{\partial B_t(0)} |\nabla u(y)| \frac{dS(y)}{t^{n-1}}\, dt$$
$$= \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$$

An integral mean value inequality

Proof

- $\displaystyle\int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta \leq \int_{B_s(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$
- Multiplying both sides by s^{n-1} and integrating over s, we get

$$\begin{aligned}\int_{B_r(0)} |u(y) - u(0)|\, dy &= \int_0^r \int_{\partial B_1(0)} |u(s\theta) - u(0)|\, d\theta s^{n-1} ds \\ &\leq \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy \int_0^r s^{n-1}\, ds \\ &= \frac{1}{n} r^n \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.\end{aligned}$$

This gives the desired integral mean value inequality.

A corollary of the integral mean value inequality

Corollary

Suppose $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ for some $p > n$. Then

$$\int_{B_r(x)} |u(y) - u(x)|\, dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} r^{\frac{n(p-1)}{p}+1} \text{ for all } B_r(x) \subset \Omega,$$

where the constant $C_{n,p}$ depends only on n and p.

Proof

- As in the previous proof, we assume without loss of generality that $x = 0$. We start with the integral mean value inequality:

$$\int_{B_r(0)} |u(y) - u(0)|\, dy \leq \frac{r^n}{n} \int_{B_r(0)} \frac{|\nabla u(y)|}{|y|^{n-1}}\, dy.$$

A corollary of the integral mean value inequality

Proof

- By Hölder's inequality this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq \frac{r^n}{n} \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_{B_r(0)} \frac{1}{|y|^{(n-1)p'}}\,dy \Big\}^{1/p'}$$
$$= C_n r^n \|\nabla u\|_{L^p(B_r(0))} \Big\{ \int_0^r s^{-(n-1)(p'-1)}\,ds \Big\}^{1/p'}.$$

- As $p > n$, we have that $p' < \frac{n}{n-1}$ and so $(n-1)(p'-1) < 1$. Hence the integral in the curly braces converges to $C_{n,p} r^{-(n-1)(p'-1)+1}$. After a simplification, this gives

$$\int_{B_r(0)} |u(y) - u(0)|\,dy \leq C_{n,p} \|\nabla u\|_{L^p(B_r(0))} r^{\frac{n}{p'}+1},$$

which is the conclusion.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Proof when $p < \infty$. The case $p = \infty$ will be dealt with later.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Suppose that (*) holds for functions in $C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. We show that this implies the theorem.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ Let $u \in W^{1,p}(\mathbb{R}^n)$. As $p < \infty$, we can find $u_m \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}$.
  - ⋆ Applying (*) to $u_m - u_j$ we have

$$\|u_m - u_j\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)} \stackrel{m,j\to\infty}{\longrightarrow} 0.$$

    This means that (u_m) is Cauchy in $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$, and hence converges in $C^{0,1-\frac{n}{p}}$ to some $u_* \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.
  - ⋆ On the other hand, as $u_m \to u$ in L^p, a subsequence of it converges a.e. in $\mathbb{R}^n$ to u.
  - ⋆ It follows that $u = u_*$ a.e. in $\mathbb{R}^n$, i.e. u has a continuous representative.

Morrey's inequality

Proof when $p < \infty$.

- Step 1: Reduction to the case $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.
  - ⋆ We may thus assume henceforth that u is continuous, so that u_m converges to u in both $W^{1,p}$ and $C^{0,1-\frac{n}{p}}$.
  - ⋆ Now, applying (*) to u_m we have

$$\|u_m\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

Sending $m \to \infty$, we hence have

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

as wanted.

Morrey's inequality

Proof when $p < \infty$.

- Step 2: Proof of the C^0 bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

  - ⋆ By triangle inequality, we have

$$|B_1(x)||u(x)| \leq \int_{B_1(x)} |u(y) - u(x)|\, dy + \int_{B_1(x)} |u(y)| dy.$$

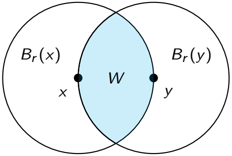
  - ⋆ By Hölder's inequality, the last integral is bounded by $C_{n,p}\|u\|_{L^p(B_1(x))}$.
  - ⋆ On the other hand, by the corollary to the integral mean value inequality, the first integral on the right hand side is bounded by $C_{n,p}\|\nabla u\|_{L^p(B_1(x))}$. The inequality (**) follows.

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*). We show that, for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, it holds that

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}|x - y|^{1-\frac{n}{p}} \text{ for all } x, y \in \mathbb{R}^n. \quad (***)$$



- ⋆ If $x = y$, there is nothing to show. Suppose henceforth that $r = |x - y| > 0$ and let $W = B_r(x) \cap B_r(y)$.
- ⋆ Let a be the average of u in W, i.e. $a = \frac{1}{|W|}\int_W u(z)\,dz$. Then

$$|u(x) - u(y)| \leq |u(x) - a| + |u(y) - a|.$$

Morrey's inequality

Proof when $p < \infty$.

- Step 3: Proof of the $C^{0,1-\frac{n}{p}}$ semi-norm bound in (*).
  - ⋆ We estimate $|u(x) - a|$ as follows:

$$|u(x) - a| \leq \frac{1}{|W|}\int_W |u(x) - u(z)|dz$$
$$\leq \frac{C_n}{r^n}\int_{B_r(x)} |u(x) - u(z)|dz.$$

By the corollary to the mean value inequality, the right hand side is bounded by $C_{n,p}\|\nabla u\|_{L^p(B_r(x))}\, r^{1-\frac{n}{p}}$. So

$$|u(x) - a| \leq C_{n,p}\|\nabla u\|_{L^p(B_r(x))}\, r^{1-\frac{n}{p}}$$

  - ⋆ Similarly, $|u(y) - a| \leq C_{n,p}\|\nabla u\|_{L^p(B_r(y))}\, r^{1-\frac{n}{p}}$.
  - ⋆ Putting these together and recalling that $r = |x - y|$, we arrive at (***).

Morrey's inequality on domain for $n < p < \infty$

Theorem (Morrey's inequality)

Suppose that $n < p < \infty$ and Ω is a bounded Lipschitz domain. Then every $u \in W^{1,p}(\Omega)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative and

$$\|u\|_{C^{0,1-\frac{n}{p}}(\Omega)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.$$

Indeed, let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then Eu has a continuous representative and

$$\begin{aligned}\|Eu\|_{C^{0,1-\frac{n}{p}}(\Omega)} &\leq \|Eu\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)}\\ &\leq C_{n,p}\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_{n,p,\Omega}\|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

An improved integral mean value inequality

Lemma

Suppose $u \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ for some $p > n$. Then, for every ball $B_r(x) \subset \mathbb{R}^n$, there holds

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \, dy.$$

Proof

- Replacing p by any $\tilde{p} \in (n, p)$, we may assume that p is finite. Then we can find $u_m \in C^\infty(B_R(0)) \cap W^{1,p}(B_R(0))$ such that $u_m \to u$ in $W^{1,p}$. Furthermore, arguing as in Step 1 in the proof of Morrey's inequality, we also have that $u_m \to u$ in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.

An improved integral mean value inequality

Proof

- $u_m \to u$ in $W^{1,p}(B_R(0))$ and in $C^{0,1-\frac{n}{p}}(\overline{B_R(0)})$.
- By the integral mean value inequality for C^1 functions, we have

$$\int_{B_r(x)} |u_m(y) - u_m(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u_m(y)|}{|y-x|^{n-1}}\, dy.$$

- The left hand side converges to $\int_{B_r(x)} |u(y) - u(x)| dy$ since $u_m \to u$ uniformly.
- The right hand side converges to $\frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy$ since $\nabla u_m \to \nabla u$ in L^p and since the function $y \mapsto \frac{1}{|y-x|^{n-1}}$ belongs to $L^{p'}$ (as noted in the proof of the corollary to the integral mean value inequality). The conclusion follows.

Morrey's inequality

Theorem (Morrey's inequality)

Assume that $n < p \leq \infty$. Then every $u \in W^{1,p}(\mathbb{R}^n)$ has a $(1 - \frac{n}{p})$-Hölder continuous representative. Furthermore there exists a constant $C_{n,p}$ such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}. \qquad (*)$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Note that when $p = \infty$ we can no longer use the previous proof, as $C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ is not dense in $W^{1,\infty}(\mathbb{R}^n)$.

Morrey's inequality

Proof when $p = \infty$.

- Suppose $u \in W^{1,\infty}(\mathbb{R}^n)$. Then $u \in W^{1,s}(B_R)$ for all $s < \infty$ and all ball B_R. By Morrey's inequality in the case of finite p, we thus have that u has a continuous representative, which we will assume to be u itself.
- By the improved integral mean value inequality, we have

$$\int_{B_r(x)} |u(y) - u(x)| dy \leq \frac{1}{n} r^n \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} dy.$$

- Step 2 and Step 3 of the proof in the case $p < \infty$ can now be repeated to get

$$|u(x)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \text{ for all } x \in \mathbb{R}^n. \qquad (**)$$

  and

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}|x-y| \text{ for all } x, y \in \mathbb{R}^n. \qquad (***)$$

Morrey's inequality

Proof when $p = \infty$.

- It follows that

$$\|u\|_{C^{0,1}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\mathbb{R}^n)}$$

and we are done.

Morrey's inequality on domains

We make a couple of remarks:

- If Ω and p are such that there exists a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ (in particular $Eu = u$ a.e. in Ω for all $u \in W^{1,p}(\Omega)$), then

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\Omega).$$

- The same proof on the whole space work on balls without establishing the existence of an extension operator. (Check this!)
- For general domains, one only has

$$W^{1,p}(\Omega) \hookrightarrow C_{loc}^{0,1-\frac{n}{p}}(\Omega).$$

(Revisit the example of the disk in $\mathbb{R}^2$ with a line segment removed.)

More on $W^{1,\infty}$

We have the following important theorem for the space $W^{1,\infty}(\Omega)$:

Theorem

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then

$$W^{1,\infty}(\Omega) = C^{0,1}(\Omega).$$

Friedrichs' inequality

Theorem (Friedrichs' inequality)

Assume that Ω is a bounded open set and $1 \leq p < \infty$. Then, there exists $C_{p,\Omega}$ such that

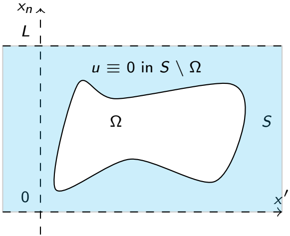
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Note that

- Only the derivatives of u appear on the right hand side.
- The function u belongs to $W_0^{1,p}(\Omega)$. The inequality is false for $u \in W^{1,p}(\Omega)$.
- By Friedrichs' inequality, when Ω is bounded, if we define $|||u||| = \|\nabla u\|_{L^p(\Omega)}$, then $||| \cdot |||$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\| \cdot \|_{W^{1,p}(\Omega)}$.
- In some text, Friedrichs' inequality is referred to as Poincaré's inequality.

Friedrichs' inequality

Proof



- We may assume that Ω is contain in the slab $S := \{(x', x_n) : 0 < x_n < L\}$.
- As usual, using the density of $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, it suffices to prove

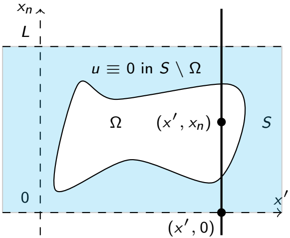
$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^p(\Omega)}$$

for $u \in C_c^\infty(\Omega)$.

- Take an arbitrary $u \in C_c^\infty(\Omega)$ and extend u by zero outside of Ω so that $u \in C_c^\infty(S)$.

Friedrichs' inequality

Proof



- Now, for every fixed x', we have

$$|u(x', x_n)| \leq \int_0^{x_n} |\partial_n u(x', t)| \, dt \leq \Big\{ \int_0^{x_n} |\partial_n u(x', t)|^p \, dt \Big\}^{1/p} x_n^{1/p'}$$
$$\leq \Big\{ \int_0^{L} |\partial_n u(x', t)|^p \, dt \Big\}^{1/p} x_n^{\frac{p-1}{p}}.$$

Friedrichs' inequality

Proof

- $|u(x', x_n)| \leq \left\{ \int_0^L |\partial_n u(x', t)|^p \, dt \right\}^{1/p} x_n^{\frac{p-1}{p}}$.
- It follows that

$$\int_0^L |u(x', x_n)|^p \, dx_n \leq \frac{1}{p} L^p \int_0^L |\partial_n u(x', t)|^p \, dt.$$

- Integrating over x' then gives

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &= \int_{\mathbb{R}^{n-1}} \int_0^L |u(x', x_n)|^p \, dx_n \, dx' \\ &\leq \frac{1}{p} L^p \int_{\mathbb{R}^{n-1}} \int_0^L |Du(x', t)|^p \, dt \, dx' = \frac{1}{p} L^p \|\nabla u\|_{L^p(\Omega)}^p. \end{aligned}$$

We are done.