BO1 History of Mathematics
Lecture XII
19th-century rigour in real analysis, continued

MT 2021 Week 6

Summary

Part 1

- Proofs of the Intermediate Value Theorem revisited
- Convergence and completeness

Part 2

- Dedekind and the continuum

Part 3

- Cantor and numbers and sets
- Where and when did sets emerge?
- Early set theory
- Set theory as a language

Part 1: Completeness

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But Bolzano assumed the existence of the limit.

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Cauchy's 2nd proof in a different context (p. 460): a numerical method for finding roots of equations — tacitly assumes that bounded monotone sequences of real numbers converge [see Lecture VII].

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  2. *A* [*non-empty*] *set of numbers bounded below has a greatest lower bound (proved by Bolzano in 1817 on the basis of (1)).*
  3. *A monotonic bounded sequence converges to a limit (taken for granted by Cauchy in 1821).*

(*Mathematics emerging*, §16.3.1.)

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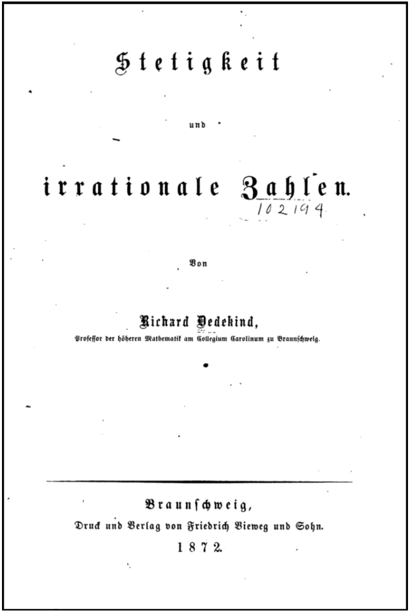
All of the above relies upon an intuitive notion of real number — so perhaps provide a formal definition of these? One that includes the idea of completeness?

Part 2: Real Numbers

Richard Dedekind (1831–1916)



Stetigkeit

und

irrationale Zahlen.

Von

Richard Dedekind,

Professor der höheren Mathematik am Collegium Carolinum zu Braunschweig.

Braunschweig,

Druck und Verlag von Friedrich Vieweg und Sohn.

1872.

Dedekind and the foundations of analysis

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- geometry as a foundation for analysis;
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Response eventually published in *Stetigkeit und irrationale Zahlen* (1872) [translated as *Continuity and irrational numbers* by Wooster Woodruff Beman, 1901]

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Geometrically, every point separates a line into two parts.

> *I find the essence of continuity in the converse, i.e., in the following principle:*
>
> *"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."*

Dedekind and continuity (2)

But Dedekind couldn't *prove* this property, so he had to take it as an axiom:

> *The assumption of this property for the line is nothing but an Axiom, through which alone we attribute continuity to the line, through which we understand continuity in the line.*

(See *Mathematics emerging*, §16.3.2.)

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Hence Dedekind cuts (or sections, from the original German Schnitt).

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Whenever, then, we have to do with a cut produced by no rational number, we create a new irrational number, which we regard as completely defined by this cut . . .

Dedekind cuts (2)

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Rechnungen mit reellen Zahlen.

Um irgend eine Rechnung mit zwei reellen Zahlen α, β auf die Rechnungen mit rationalen Zahlen zurückzuführen, kommt es nur darauf, aus den Schnitten (A_1, A_2) und (B_1, B_2), welche durch die Zahlen α und β im Systeme R hervorgebracht werden, den Schnitt (C_1, C_2) zu definiren, welcher dem Rechnungsresultate γ entsprechen soll. Ich beschränke mich hier auf die Durchführung des einfachsten Beispieles, der Addition.

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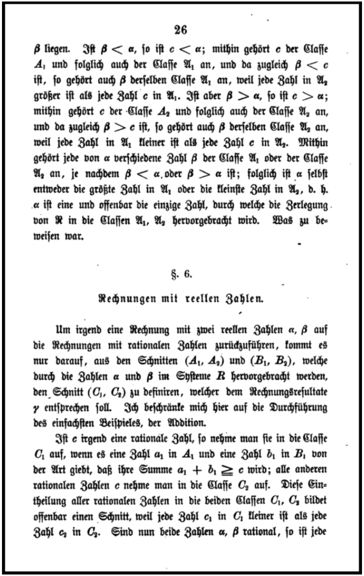

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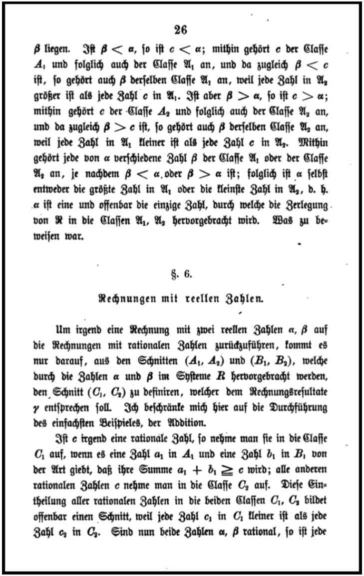

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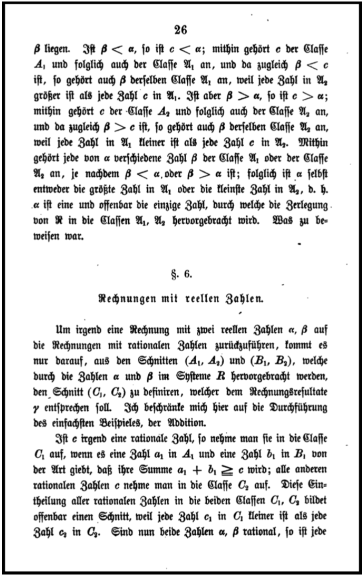

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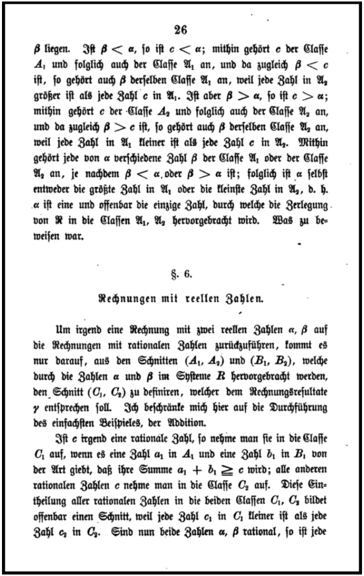
Dedekind showed how to add two cuts, and how to use them in limiting arguments — but did little else with them.

Significance: a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence

Dedekind cuts (2)



26

β liegen. Ist $\beta < \alpha$, so ist $c < \alpha$; mithin gehört c der Classe A_1 und folglich auch der Classe $\mathfrak{A}_1$ an, und da zugleich $\beta < c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_1$ an, weil jede Zahl in $\mathfrak{A}_2$ größer ist als jede Zahl c in $\mathfrak{A}_1$. Ist aber $\beta > \alpha$, so ist $c > \alpha$; mithin gehört c der Classe A_2 und folglich auch der Classe $\mathfrak{A}_2$ an, und da zugleich $\beta > c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_2$ an, weil jede Zahl in $\mathfrak{A}_1$ kleiner ist als jede Zahl c in $\mathfrak{A}_2$. Mithin gehört jede von α verschiedene Zahl β der Classe $\mathfrak{A}_1$ oder der Classe $\mathfrak{A}_2$ an, je nachdem $\beta < \alpha$ oder $\beta > \alpha$ ist; folglich ist α selbst entweder die größte Zahl in $\mathfrak{A}_1$ oder die kleinste Zahl in $\mathfrak{A}_2$, d. h. α ist eine und offenbar die einzige Zahl, durch welche die Zerlegung von $\mathfrak{R}$ in die Classen $\mathfrak{A}_1$, $\mathfrak{A}_2$ hervorgebracht wird. Was zu beweisen war.

§. 6.

Rechnungen mit reellen Zahlen.

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Significance: a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence
- setting the foundations of analysis onto a sound logical basis.

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A good modern (historically sensitive) account can be found in: Leo Corry, *A brief history of numbers*, OUP, 2015, §10.6.

Other approaches

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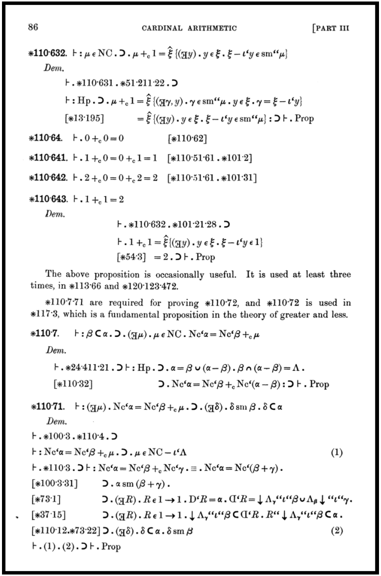
Later constructions by many mathematicians and philosophers — such as

- Carl Johannes Thomae, 1880, 1890;
- Giuseppe Peano, 1889, 1891;
- Gottlob Frege, 1884, 1893, 1903;
- Otto Hölder, 1901;
- ...

Extreme formalism



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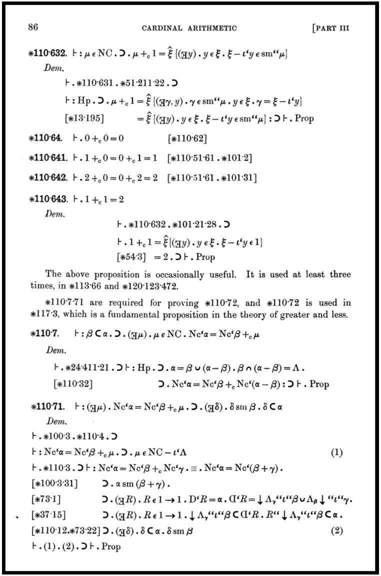
Alfred North Whitehead and Bertrand Russell, *Principia mathematica*, 3 vols., Cambridge University Press, 1910, 1912, 1913

Extreme formalism



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Vol. II, p. 86: $1 + 1 = 2$

Extreme formalism

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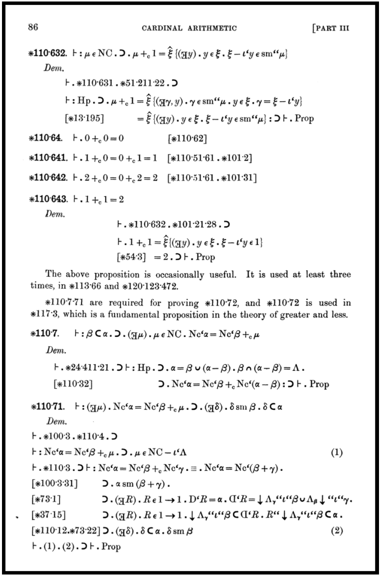

86 CARDINAL ARITHMETIC [PART III

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Vol. II, p. 86: $1 + 1 = 2$

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NB. This is not the source of our axioms for the reals.

Part 3: Sets

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An idea that emerged as central to Dedekind's work: that of a set

In fact, naive notions of sets had already appeared all but unremarked earlier in the nineteenth century

- as Gauss' classes, orders, genera (of binary quadratic forms with integer coefficients) [see Lecture XIV];
- as Galois' groupes (of permutations and of substitutions);
- as Cauchy's systèmes (of substitutions);
- as Dedekind's Zahlkörpern (of algebraic numbers).

This is by no means an exhaustive list of examples; see *Mathematics emerging*, §18.2 for others.

Formalisation of the concept of a set

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> *By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.*

Cantor and the continuum

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Cantor's first great insight regarding sets (1873): infinite sets can have different sizes.

Cantor's first proof that the continuum is uncountable

Proposition: Given any sequence of real numbers $\omega_1, \omega_2, \omega_3, \ldots$ and any interval $[\alpha, \beta]$, there is a real number in $[\alpha, \beta]$ that is not contained in the given sequence.

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Proof proceeds by construction of a sequence of nested intervals $[\alpha, \beta] \supseteq [\alpha_1, \beta_1] \supseteq [\alpha_2, \beta_2] \supseteq [\alpha_3, \beta_3] \supseteq \cdots$. Cantor considered the different cases where the sequence terminates or does not, but in all instances he constructed a real number in the interval that does not lie in the original sequence.

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Next suppose that the continuum is countable, i.e., that the real numbers may be listed $\omega_1, \omega_2, \omega_3, \ldots$. But then there is a real number in any interval $[\alpha, \beta]$ that does not belong to this list — a contradiction.

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The more famous diagonal argument came later (1891).

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Cantor to Dedekind (1877): there is a one-to-one correspondence between a line and the plane — "Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")

Cantor's *Mengenlehre*

Developed at the end of the nineteenth century (1878–1897): a general theory of sets and of transfinite numbers — infinite cardinals (e.g., $\#\mathbb{N} = \aleph_0$, $\#\mathbb{R} = c$), transfinite ordinals, ...

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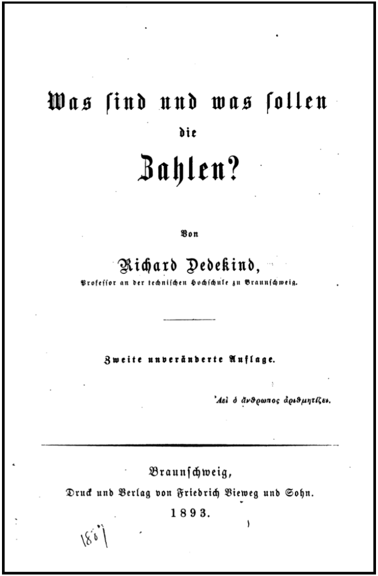
Further: $\#\mathscr{P}(\mathbb{N}) = \#\mathbb{R}$, or $2^{\aleph_0} = c$

Was sind und was sollen die Zahlen?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die
Zahlen?

Von

Richard Dedekind,
Professor an der technischen Hochschule zu Braunschweig.

Zweite unveränderte Auflage.

Ἀεὶ ὁ ἄνθρωπος ἀριθμητίζει.

Braunschweig,
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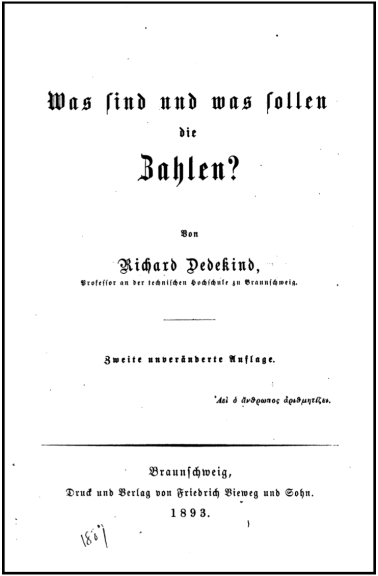

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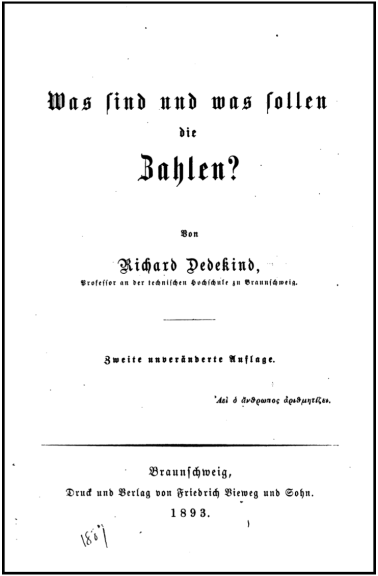

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Contains, amongst other things:

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- an axiomatisation of the natural numbers (soon simplified by Peano).

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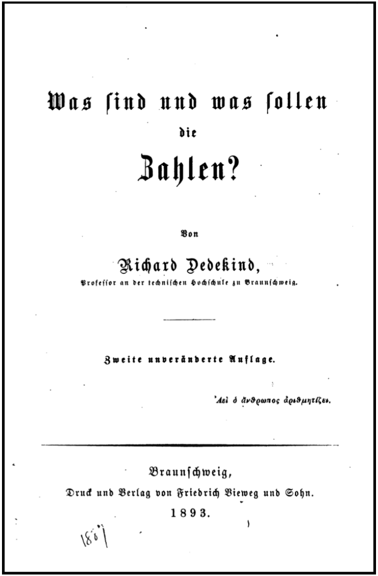

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Also includes a definition of a function as a mapping between sets (p. 6):

"By a mapping of a system S we understand a law according to which every determinate element s of S is associated with a determinate thing which is called the *image* of s and is denoted by $\phi(s)$ ..."

Was sind und was sollen die Zahlen?

Extract from William Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, OUP, 1996, vol. II, p. 790:

> *The title of Dedekind's paper is subtle: rigidly translated it asks 'What are, and what ought to be, the numbers?' But sollen here carries several senses—among them, 'What is the best way to regard the numbers?'; 'What is the function of numbers?; 'What are numbers supposed to be?'. But perhaps Dedekind's title is famous enough to be left in the original.*

W. W. Beman translated the essay under the title *The nature and meaning of numbers* (1901).

Was sind und was sollen die Zahlen?

(wegen der Aehnlichkeit von φ) auch a' und jedes Element u' verschieden von a und folglich in T enthalten sein; mithin ist $\psi(T) \, 3 \, T$, und da T endlich ist, so muß $\psi(T) = T$, also $\mathfrak{M}(a', U') = T$ sein. Hieraus folgt aber (nach 15)

$$\mathfrak{M}(a', a, U') = \mathfrak{M}(a, T),$$

d. h. nach dem Obigen $S' = S$. Also ist auch in diesem Falle der erforderliche Beweis geführt.

§. 6.

Einfach unendliche Systeme. Reihe der natürlichen Zahlen.

71. Erklärung. Ein System N heißt einfach unendlich, wenn es eine solche ähnliche Abbildung φ von N in sich selbst giebt, daß N als Kette (44) eines Elementes erscheint, welches nicht in $\varphi(N)$ enthalten ist. Wir nennen dies Element, das wir im Folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und sagen zugleich, das einfach unendliche System N sei durch diese Abbildung φ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§. 4), so besteht mithin das Wesen eines einfach unendlichen Systems N in der Existenz einer Abbildung φ von N und eines Elementes 1, die den folgenden Bedingungen α, β, γ, δ genügen:

α. $N' \, 3 \, N$.

β. $N = 1_0$.

γ. Das Element 1 ist nicht in N' enthalten.

δ. Die Abbildung φ ist ähnlich.

Offenbar folgt aus α, γ, δ, daß jedes einfach unendliche System N wirklich ein unendliches System ist (64), weil es einem echten Theile N' seiner selbst ähnlich ist.

72. Satz. In jedem unendlichen Systeme S ist ein einfach unendliches System N als Theil enthalten.

Written in an explicitly set-theoretic language

(But with slightly different notation from ours.)

For a summary, see: Kathryn Edwards, ‘Richard Dedekind (1831–1916)’, *Mathematics Today* **52**(1) (Feb 2016) 212–215

Set theory in our lives

Set theory as an effective language for mathematics:

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Set theory as an effective language for mathematics:

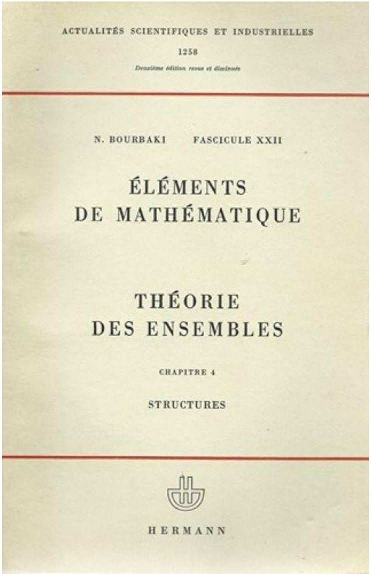
- Set-builder notation

Set theory in our lives

Set theory as an effective language for mathematics:

- Set-builder notation
- Unification of ideas concerning functions and relations

Nicolas Bourbaki (1934–????)



Collective of French mathematicians who set out to reformulate mathematics on extremely formal, abstract, structural lines — the language of sets has a significant role to play.

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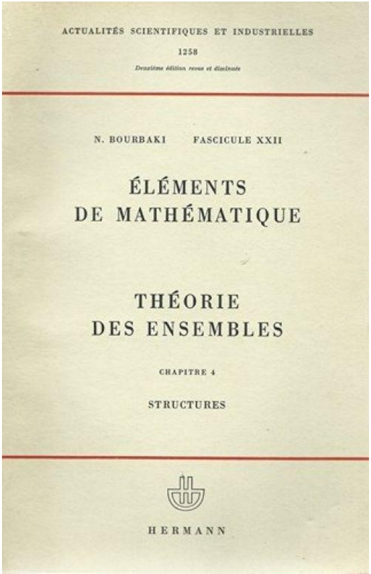

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Association des collaborateurs de Nicolas Bourbaki

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- Tom Lehrer song, New Math
- Much debate — now usually regarded as a passing fad

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- The concept of set emerged at about the same time as the modern concept of real number, 1870–1890.
- This coincidence is no coincidence.

Further reading on the development of analysis