The Scattering Transform: a deterministic transform with depth

THEORIES OF DEEP LEARNING: C6.5, LECTURE / VIDEO 12
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Scattering Transform (Mallat 12')

Repeated application of deterministic transforms

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The Scattering Transform repeatedly applied a deterministic wavelet transform followed by $\sigma(x) = |x|$ as nonlinear activation

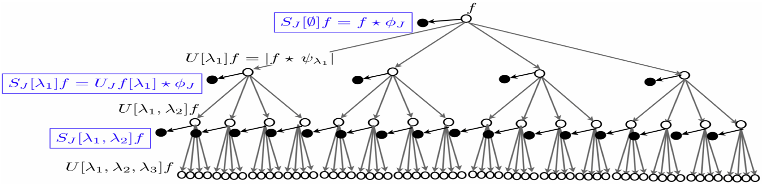


Figure 1: A scattering propagator U_J applied to f computes each $U[\lambda_1]f = |f \star \psi_{\lambda_1}|$ and outputs $S_J[\emptyset]f = f \star \phi_{2^J}$. Applying U_J to each $U[\lambda_1]f$ computes all $U[\lambda_1, \lambda_2]f$ and outputs $S_J[\lambda_1] = U[\lambda_1] \star \phi_{2^J}$. Applying iteratively U_J to each $U[p]f$ outputs $S_J[p]f = U[p]f \star \phi_{2^J}$ and computes the next path layer.

Depth allows the transform to become increasingly invariant to translation and small diffeomorphisms.

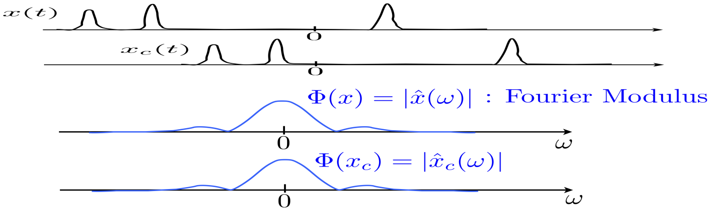
`https://arxiv.org/pdf/1101.2286.pdf`

Classification as learning invariance (Mallat '13)

Projecting out invariants not needed for classification

Invariance to translations $x_c(t) = x(t-c)$

$$\forall c \in \mathbb{R} \quad , \quad \Phi(x_c) = \Phi(x) \; .$$



Lipschitz stable to deformations $x_\tau(t) = x(t - \tau(t))$

small deformations of $x \implies$ small modifications of $\Phi(x)$

$$\forall \tau \quad , \quad \|\Phi(x_\tau) - \Phi(x)\| \le C \underbrace{\sup_t |\nabla \tau(t)|}_{\text{deformation size}} \, \|x\| \; .$$

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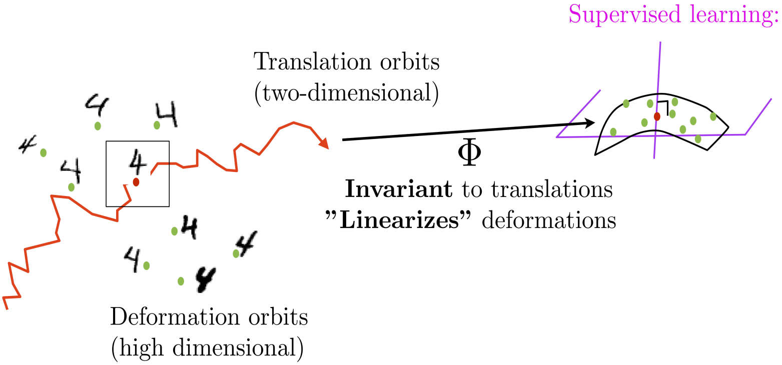
- Specific deformation invariance must be learned.



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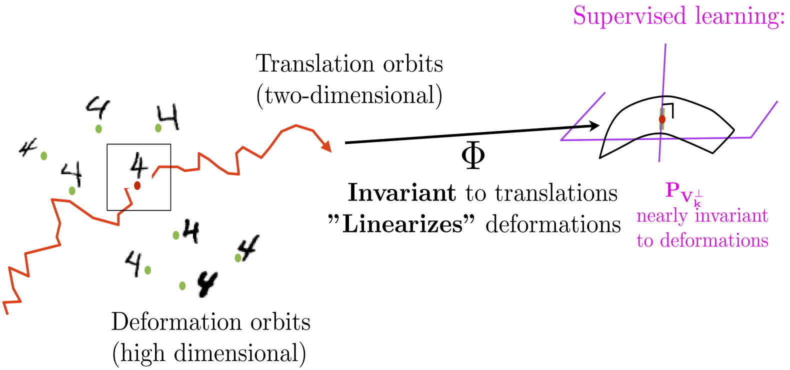
Linearising deformations (Mallat '13)

Projecting out invariants not needed for classification

- Specific deformation invariance must be learned.



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Linearising deformations (Mallat '13)

Projecting out invariants not needed for classification

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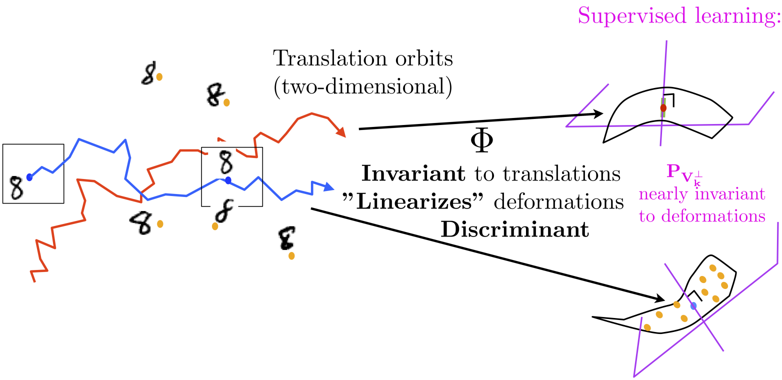

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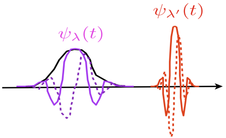
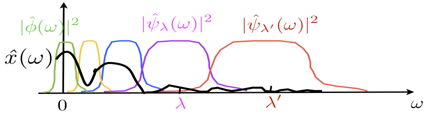
Wavelet Transform as frequency tiling (Mallat '13)

Wavelets decompose function into local frequency information

- Complex wavelet: $\psi(t) = \psi^a(t) + i\,\psi^b(t)$
- Dilated: $\psi_\lambda(t) = 2^{-j}\,\psi(2^{-j}t)$ with $\lambda = 2^{-j}$.





- Wavelet transform: $x \star \psi_\lambda(t) = \int x(u)\,\psi_\lambda(t-u)\,du$

$$Wx = \begin{pmatrix} x \star \phi(t) \\ x \star \psi_\lambda(t) \end{pmatrix}_{t,\lambda}$$

Unitary: $\|Wx\|^2 = \|x\|^2$.

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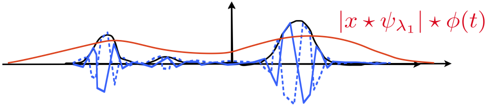
Modulus and averaging in wavelet domain (Mallat '13)

Smoothing to identify discontinuities and have energy decay



- The modulus $|x \star \psi_{\lambda_1}|$ is a regular envelop
- The average $|x \star \psi_{\lambda_1}| \star \phi(t)$ is invariant to small translations relatively to the support of ϕ.
- Full translation invariance at the limit:

$$\lim_{\phi \to 1} |x \star \psi_{\lambda_1}| \star \phi(t) = \int |x \star \psi_{\lambda_1}(u)|\, du = \|x \star \psi_{\lambda_1}\|_1$$

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Second layer of the scattering transform (Mallat '13)

Increased smoothness with depth

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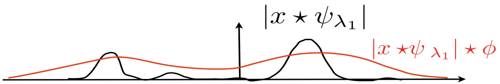

- The high frequencies of $|x \star \psi_{\lambda_1}|$ are in wavelet coefficients:

$$W|x \star \psi_{\lambda_1}| = \begin{pmatrix} |x \star \psi_{\lambda_1}| \star \phi(t) \\ |x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}(t) \end{pmatrix}_{t,\lambda_2}$$

- Translation invariance by time averaging the amplitude:

$$\forall \lambda_1, \lambda_2, \quad ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi(t)$$

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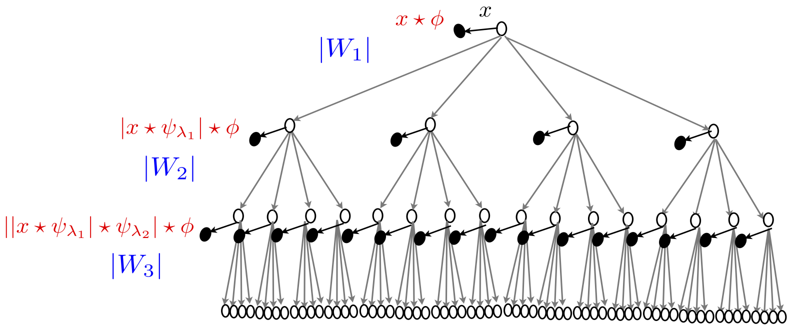
Scattering transform (Mallat '13)

Lipshitz continuous, inputs contract to one another



- Cascade of contractive operators

$$\| |W_k| x - |W_k| x' \| \leq \| x - x' \| \quad \text{with} \quad \| |W_k| x \| = \| x \| \; .$$

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$$Sx = \left(\begin{array}{c} x \star \phi\ (u) \\ |x \star \psi_{\lambda_1}| \star \phi(u) \\ ||x \star \psi\ _{\lambda_1}| \star \psi_{\lambda_2}| \star \phi(u) \\ |||x \star \psi\ _{\lambda_2}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi(u) \\ \dots \end{array} \right)_{u,\lambda_1,\lambda_2,\lambda_3,\dots}$$

Theorem: *For appropriate wavelets, a scattering is*

contractive $\|Sx - Sy\| \leq \|x - y\|$

preserves norms $\|Sx\| = \|x\|$

stable to deformations $x_\tau(t) = x(t - \tau(t))$

$$\|Sx - Sx_\tau\| \leq C \sup_t |\nabla \tau(t)| \, \|x\|$$

$\Rightarrow$ linear discriminative classification from $\Phi x = Sx$

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Scattering Transform: energy decay (Mallat 12')

The transform can be truncated stably

Lemma

For suitably chosen wavelet transforms (see Theorem 2.6 in footnote) then for all $f \in L^2(\mathbb{R}^d)$

$$\lim_{m\to\infty} \|U[\Lambda_J^m]f\|^2 = \lim_{m\to\infty} \sum_{n=m}^{\infty} \|S_J[\Lambda_J^n]f\|^2 = 0$$

where $U[\lambda]f = |f \star \psi_\lambda|$ and $S_J[\lambda]f = \phi_j \star U[\lambda]f$ and $\|S_J[P_J]f\| = \|f\|$. Morevover, for all $c \in \mathbb{R}^d$

$$\lim_{J\to\infty} \|S_J[P_J]f - S_J[P_J]L_c f\| = 0$$

where $L_c f = f(x-c)$ is the translation operator.

https://arxiv.org/pdf/1101.2286.pdf

Scattering Transform: energy decay (Mallat 13')

Energy decay on CalTech-101

TABLE 1

Percentage of Energy $\sum_{p \in \mathcal{P}_\downarrow^m} \|S[p]x\|^2 / \|x\|^2$ of Scattering Coefficients on Frequency-Decreasing Paths of Length m, Depending upon J

J	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m \leq 3$
1	95.1	4.86	-	-	-	99.96
2	87.56	11.97	0.35	-	-	99.89
3	76.29	21.92	1.54	0.02	-	99.78
4	61.52	33.87	4.05	0.16	0	99.61
5	44.6	45.26	8.9	0.61	0.01	99.37
6	26.15	57.02	14.4	1.54	0.07	99.1
7	0	73.37	21.98	3.56	0.25	98.91

These average values are computed on the Caltech-101 database, with zero mean and unit variance images.

https://www.di.ens.fr/data/publications/papers/pami-final.pdf

Scattering Transform: MNIST classification (Mallat 13')

Accuracy on MNIST based on training size

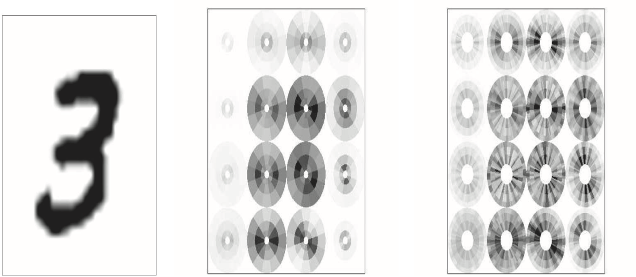
TABLE 4
Percentage of Errors of MNIST Classifiers, Depending on the Training Size

Training size	x PCA	x SVM	Wind. Four. PCA	Wind. Four. SVM	Scat. $\overline{m}=1$ PCA	Scat. $\overline{m}=1$ SVM	Scat. $\overline{m}=2$ PCA	Scat. $\overline{m}=2$ SVM	Conv. Net.
300	14.5	15.4	7.35	7.4	5.7	8	**4.7**	5.6	7.18
1000	7.2	8.2	3.74	3.74	2.35	4	**2.3**	2.6	3.21
2000	5.8	6.5	2.99	2.9	1.7	2.6	**1.3**	1.8	2.53
5000	4.9	4	2.34	2.2	1.6	1.6	**1.03**	1.4	1.52
10000	4.55	3.11	2.24	1.65	1.5	1.23	0.88	1	**0.85**
20000	4.25	2.2	1.92	1.15	1.4	0.96	0.79	**0.58**	0.76
40000	4.1	1.7	1.85	0.9	1.36	0.75	0.74	**0.53**	0.65
60000	4.3	1.4	1.80	0.8	1.34	0.62	0.7	**0.43**	0.53

https://www.di.ens.fr/data/publications/papers/pami-final.pdf

Scattering Transform: MNIST digit 3 (Mallat 13')

Example of energy in a scattering transform



(a) (b) (c)

Fig. 7. (a) Image $X(u)$ of a digit "3." (b) Arrays of windowed scattering coefficients $S[p]X(u)$ of order $m = 1$, with u sampled at intervals of $2^J = 8$ pixels. (c) Windowed scattering coefficients $S[p]X(u)$ of order $m = 2$.

https://www.di.ens.fr/data/publications/papers/pami-final.pdf