C4.3 Functional Analytic Methods for PDEs
Lecture 11

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MT 2021

In the last 6 lectures

- Sobolev spaces and their properties

This lecture

- Linear elliptic equations of second order
- Classical and weak solutions
- Energy estimates
- First existence theorem

The equation of interest

- We will consider the equation

$$Lu := -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  where
  - ⋆ Ω is a domain in $\mathbb{R}^n$, which frequently has Lipschitz regularity or better,
  - ⋆ $u : \Omega \to \mathbb{R}$ is the unknown,
  - ⋆ $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are given coefficients,
  - ⋆ $f, g_i : \Omega \to \mathbb{R}$ are given sources.
- Equation (*) is said to be in divergence form. It can be written in more compact form:

$$Lu = -\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu = f + \mathrm{div} g$$

  where
  - ⋆ $a = (a_{ij})$ is an $n \times n$ matrix,
  - ⋆ $b = (b_i)$ and $g = (g_i)$ are (column) vectors.

Divergence vs non-divergence form

- To dispel confusion, we note that we will not consider the equation

$$-a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega, \qquad (**)$$

which is also of importance. The equation (**) is said to be in non-divergence form.
To treat (**), we will need some preparation different from what we have had so far.

Structural assumptions

We make the following assumptions:

- The coefficients $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ belong to $L^\infty(\Omega)$.
- The coefficients a_{ij} is *symmetric*, i.e. $a_{ij} = a_{ji}$.
- The coefficients a_{ij} is *uniformly elliptic* – this will be defined on the next slide.

Ellipticity

Definition

Let $a = (a_{ij}) : \Omega \to \mathbb{R}^{n\times n}$ be symmetric and have measurable entries.

- a is *elliptic* if

$$a_{ij}(x)\xi_i\,\xi_j \geq 0 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

  (In other words, a is non-negative definite a.e. in Ω.)
- a is *strictly elliptic* if there exists $\lambda > 0$ such that

$$a_{ij}(x)\xi_i\,\xi_j \geq \lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

- a is *uniformly elliptic* if there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\,\xi_j \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

Examples

Two simplistic but important examples:

- $a_{ij} = \delta_{ij}$ in all of Ω.
- $a_{ij} = k(x)\delta_{ij}$ where $k = k_1\chi_A + k_2\chi_{\Omega\setminus A}$ for some subset A of Ω and some constants $k_1, k_2 > 0$.

The Dirichlet boundary value problem

We will write $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ to mean that

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu.$$

The Dirichlet boundary value problem for L asks to find a function u satisfying

$$\left\{\begin{array}{rcll} Lu &=& f + \partial_i g_i & \text{in } \Omega, \\ u &=& u_0 & \text{on } \partial\Omega. \end{array}\right. \tag{BVP}$$

where

- $\star$ f and g are given sources,
- $\star$ u_0 is given boundary data.

Classical solutions

$$L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c.$$

$$\begin{cases} Lu &= f + \partial_i g_i \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega. \end{cases} \tag{BVP}$$

Definition

Suppose $a \in C^1(\Omega), b, c \in C(\Omega)$. For a given $f \in C(\Omega)$, $g \in C^1(\Omega)$ and $u_0 \in C(\partial\Omega)$, a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is called a *classical solution to* the Dirichlet boundary value problem (BVP) if it satisfies (BVP) in the usual sense.

- We saw in the first lecture that the notion of classical solutions is insufficient for our need.

An observation

- Suppose $a \in C^1(\Omega)$, $b, c \in C(\Omega)$, $f \in C(\Omega)$ and $g \in C^1(\Omega)$. Suppose $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega. \qquad (*)$$

- If $\varphi \in C_c^\infty(\Omega)$ is a test function, then

$$\int_\Omega (Lu)\,\varphi\,dx = \int_\Omega \Big[a_{ij}\partial_j u\partial_i\varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx$$

and

$$\int_\Omega [f + \partial_i g_i]\,\varphi\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx.$$

- Therefore, for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \Big[a_{ij}\partial_j u\partial_i\varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx. \qquad (\diamondsuit)$$

An observation

- Conversely, if u is such that $(\diamondsuit)$ holds for all $\varphi \in C_c^\infty(\Omega)$, then by reversing the argument, we have

$$\int_\Omega (Lu)\,\varphi\,dx = \int_\Omega [f + \partial_i g_i]\,\varphi\,dx \text{ for all } \varphi \in C_c^\infty(\Omega).$$

  This implies $Lu = f + \partial_i g_i$ in Ω, i.e. u satisfies (*).
- We conclude that $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  if and only if u satisfies

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i\varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx \qquad (\diamondsuit)$$

  for all $\varphi \in C_c^\infty(\Omega)$.

An observation

- We conclude that $u \in C^2(\Omega)$ satisfies

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f + \partial_i g_i \text{ in } \Omega \qquad (*)$$

  if and only if u satisfies

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i\varphi + b_i\partial_i u\varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx \qquad (\Diamond)$$

  for all $\varphi \in C_c^\infty(\Omega)$.
- Key: While the formulation (*) requires u to be twice differentiable, the formulation $(\Diamond)$ requires u to be only once differentiable.

Weak solutions

Definition

Let $a, b, c \in L^\infty(\Omega)$ and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

- Suppose $f \in L^2(\Omega)$, $g \in L^2(\Omega)$.
  We say that $u \in H^1(\Omega)$ is a *weak solution* (or *generalized solution*) to the equation

$$Lu = f + \partial_i g_i \text{ in } \Omega \tag{*}$$

  if

$$\int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i \partial_i u \varphi + cu\varphi\Big]\,dx = \int_\Omega [f\varphi - g_i\partial_i\varphi]\,dx \qquad (\diamondsuit)$$

  holds for all $\varphi \in H_0^1(\Omega)$.
  When this holds, we also say that u satisfies (*) in the weak sense.

Weak solutions

Definition

Let $a, b, c \in L^\infty(\Omega)$ and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

- Suppose that $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$. We say that $u \in H^1(\Omega)$ is a *weak solution* (or *generalized solution*) to the Dirichlet boundary value problem

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{BVP}$$

if $Lu = f + \partial_i g_i$ in Ω in the weak sense and if $u - u_0 \in H^1_0(\Omega)$.

Weak solutions

- It is convenient to introduce the bilinear form $B(\cdot,\cdot)$:

$$B(u,v) = \int_\Omega [a_{ij}\partial_j u \partial_i v + b_i \partial_i u v + cuv]\, dx \qquad u, v \in H^1(\Omega).$$

  B is called the bilinear form associated with the operator L.
- Then $u \in H^1(\Omega)$ satisfies (*) in the weak sense if

$$B(u,\varphi) = \langle f, \varphi\rangle - \langle g_i, \partial_i \varphi\rangle \text{ for all } \varphi \in H^1_0(\Omega),$$

  where $\langle \cdot,\cdot\rangle$ denotes the inner product of $L^2(\Omega)$.

Energy estimate

Theorem (Energy estimates)

Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ and B is its associated bilinear form. Then there exists some large constant $C > 0$ such that

$$|B(u,v)| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

$$\frac{\lambda}{2}\|u\|^2_{H_1(\Omega)} \leq B[u,u] + C\|u\|^2_{L^2(\Omega)}.$$

Here λ is the constant appearing in the definition of ellipticity of a.

Energy estimate

Proof

- The first estimate is clear from the definition of B and Cauchy-Schwarz's inequality:

$$
\begin{aligned}
|B(u,v)| &\leq \int_\Omega \Big[|a_{ij}||\partial_j u||\partial_i v| + |b_i||\partial_i u||v| + |c||u||v|\Big]\,dx \\
&\leq \|a\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2} \\
&\qquad + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} \\
&\leq C\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}
$$

Energy estimate

Proof

- For the second estimate, we start by estimating the lower order term in the same fashion while leaving the highest order term untouched:

$$
\begin{aligned}
B(u,u) &\geq \int_\Omega \Big[a_{ij}\partial_j u \partial_i u - |b_i||\partial_i u||u| - |c||u|^2\Big]\,dx \\
&\geq \int_\Omega a_{ij}\partial_j u \partial_i u\,dx \\
&\qquad\qquad - \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^2} - \|c\|_{L^\infty}\|u\|_{L^2}^2.
\end{aligned}
$$

- The leading term is treated using the ellipticity condition:

$$a_{ij}\partial_j u \partial_i u \geq \lambda |\nabla u|^2.$$

Energy estimate

Proof

- We thus have

$$B(u,u) \geq \lambda\|\nabla u\|_{L^2}^2 - \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^2} - \|c\|_{L^\infty}\|u\|_{L^2}^2.$$

- Using the inequality $xy \leq \frac{\lambda}{2}x^2 + \frac{1}{2\lambda}y^2$, we can absorb the quantity $\|\nabla u\|_{L^2}$ in the second term on the right hand side to the first term:

$$\begin{aligned} B(u,u) &\geq \lambda\|\nabla u\|_{L^2}^2 - \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2\lambda}\|b\|_{L^\infty}^2\|u\|_{L^2}^2 - \|c\|_{L^\infty}\|u\|_{L^2}^2 \\ &= \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^2. \end{aligned}$$

L as an operator on $H^1(\Omega)$

Corollary

Suppose that $a, b, c \in L^\infty(\Omega)$*, a is uniformly elliptic,*
$L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.
For every $u \in H^1(\Omega)$*, define a map* $Lu : H_0^1(\Omega) \to \mathbb{R}$ *by*

$$(Lu)(\varphi) = B(u, \varphi) \text{ for all } \varphi \in H_0^1(\Omega).$$

Then $Lu : H_0^1(\Omega) \to \mathbb{R}$ *is bounded linear, i.e.*
$Lu \in (H_0^1(\Omega))^* =: H^{-1}(\Omega)$.
Furthermore, L is a bounded linear map from $H^1(\Omega)$ *into* $H^{-1}(\Omega)$.

L as an operator on $H^1(\Omega)$

Proof

- Linearity if clear. By the energy estimate, $|(Lu)(\varphi)| \leq C\|u\|_{H^1}\|\varphi\|_{H^1}$ and so Lu belongs to $H^{-1}(\Omega)$.
- Furthermore, we have

$$\|Lu\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|_{H^1} \leq 1} |Lu(\varphi)| \leq C\|u\|_{H^1}.$$

This means $L \in \mathscr{L}(H^1(\Omega), H^{-1}(\Omega))$.

Weak sense vs H^{-1} sense

Corollary

u is a weak solution to (*) if and only if $Lu = f + \partial_i g_i$ as elements of $H^{-1}(\Omega)$.

Here $f + \partial_i g_i$ is viewed as an element of $H^{-1}(\Omega)$ by letting

$$(f + \partial_i g_i)(\varphi) = \int_\Omega [f\varphi - g_i \partial_i \varphi]\, dx.$$

$W^{1,p}$ solutions

Remark

One can similarly define a notion of $W^{1,p}$ solutions to (*) and (BVP) using $p \neq 2$. The treatment for these type of solutions is beyond the scope of this course.

An existence theorem

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, *a is uniformly elliptic,* $c \geq 0$ *a.e. in* Ω, *and* $L = -\partial_i(a_{ij}\partial_j) + c$ *(i.e.* $b \equiv 0$*). Then for every* $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ *and* $u_0 \in H^1(\Omega)$, *the Dirichlet boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{BVP}$$

has a unique weak solution $u \in H^1(\Omega)$.

An existence theorem

The above theorem is a consequence of the following statement:

Theorem

Suppose that $a, c \in L^\infty(\Omega)$, a is uniformly elliptic, $c \geq 0$ a.e. in Ω, and $L = -\partial_i(a_{ij}\partial_j) + c$ (i.e. $b \equiv 0$). Then $L|_{H_0^1(\Omega)}$ is a bijection from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

Indeed, if we let $L^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ be the inverse of $L|_{H_0^1(\Omega)}$, then the unique solution to (BVP) is given by

$$u = u_0 + L^{-1}(-Lu_0 + f + \partial_i g_i).$$

An existence theorem

First proof: Riesz representation theorem.

- Observe that the bilinear form associated with L is positive in $H_0^1(\Omega)$:

$$B(u,u) = \int_\Omega \Big[a_{ij}\partial_j u \partial_i u + cu^2\Big]\,dx$$
$$\geq \lambda \|\nabla u\|_{L^2}^2 \geq \frac{1}{C}\|u\|_{H^1}^2 \text{ for all } u \in H_0^1(\Omega).$$

  Hence $B(\cdot,\cdot)$ defines an inner product on $H_0^1(\Omega)$, which is equivalent to the standard inner product of $H_0^1(\Omega)$.

- Thus, by the Riesz representation theorem, for every $T \in H^{-1}(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$B(u,v) = Tv \text{ for all } v \in H_0^1(\Omega).$$

  But this means precisely that $Lu = T$. We conclude that $L|_{H_0^1(\Omega)}$ is a bijection from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.