FIRST PUBLIC EXAMINATION

Honour Moderations in Mathematics

PAPER II: ANALYSIS I

Thursday, 25 June 1970, 2.30 p.m.

1. Let $f\colon [a,b] \to \mathbf{R}$ be a continuous real valued function defined on the closed interval $[a,b]$. Prove that f attains a maximum value.

Suppose, further, that $f(a) = f(b) = 0$ and that $f'(x)$, the derivative of f at x, exists for $a < x < b$. Prove that there exists θ, $a < \theta < b$, such that $f'(\theta) = 0$.

By considering
$$\begin{vmatrix} \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \\ 1 & 1 & 1 \end{vmatrix}$$
or otherwise, prove that when ϕ, ψ are functions on $[a,b]$ satisfying certain conditions, which must be stated, then
$$\{\phi(b)-\phi(a)\}\psi'(\theta) = \{\psi(b)-\psi(a)\}\phi'(\theta)$$
for some θ, $a < \theta < b$.

2. Show that
$$\sum_{r=1}^{\infty} \left\{ \int_r^{r+1} \log x\, dx - \tfrac{1}{2}[\log r + \log(r+1)] \right\}$$
is a convergent series.

Hence, or otherwise, show the existence of
$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}}.$$

3. Let $I_p = \int_0^{\frac{1}{2}\pi} (\sin x)^p\, dx$.

(i) Show that $I_{2n-1} > I_{2n} > I_{2n+1}$ for $n \geqslant 1$.

(ii) Show, by integration by parts, that $I_p = \dfrac{p-1}{p} I_{p-2}$ for $p \geqslant 2$. Deduce that
$$I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\frac{\pi}{2}$$
$$I_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}.$$

(iii) From (i) and (ii) deduce that $\lim_{n\to\infty} \dfrac{I_{2n}}{I_{2n+1}} = 1$ and so prove that
$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{[(2n)!]^2(2n+1)}.$$

(iv) Assume that the following limit exists and use (iii) to prove that
$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} = \sqrt{2\pi}.$$

Turn over.