C4.3 Functional Analytic Methods for PDEs
Lecture 12

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In the last lecture

- Linear elliptic equations of second order
- Classical and weak solutions
- Energy estimates
- First existence theorem: Riesz representation theorem

This lecture

- First existence theorem: Direct method of the calculus of variation.
- Second existence theorem: Fredholm alternative.

An existence theorem

Theorem

Suppose that $a, c \in L^\infty(\Omega)$*,* a *is uniformly elliptic,* $c \geq 0$ *a.e. in* Ω*, and* $L = -\partial_i(a_{ij}\partial_j) + c$ *(i.e.* $b \equiv 0$*). Then for every* $f \in L^2(\Omega)$*,* $g \in L^2(\Omega)$ *and* $u_0 \in H^1(\Omega)$*, the Dirichlet boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{BVP}$$

has a unique weak solution $u \in H^1(\Omega)$*.*

$$\Updownarrow$$

Theorem

Suppose that $a, c \in L^\infty(\Omega)$*,* a *is uniformly elliptic,* $c \geq 0$ *a.e. in* Ω*, and* $L = -\partial_i(a_{ij}\partial_j) + c$ *(i.e.* $b \equiv 0$*). Then* $L|_{H_0^1(\Omega)}$ *is a bijection from* $H_0^1(\Omega)$ *into* $H^{-1}(\Omega)$*.*

An existence theorem

First proof: Riesz representation theorem.

- The equation $Lu = T$ with $T \in H^{-1}(\Omega)$ is equivalent to

$$B(u, v) = Tv \text{ for all } v \in H_0^1(\Omega).$$

- The bilinear form $B(\cdot, \cdot)$ defines an inner product on $H_0^1(\Omega)$, which is equivalent to the standard inner product of $H_0^1(\Omega)$. The conclusion is reached using the Riesz representation theorem.

Second proof: Direct method of the calculus of variation.

We'll use the fact that $H_0^1(\Omega)$ is weakly closed in $H^1(\Omega)$. This is a consequence of the following general theorem:

Theorem (Mazur)

Let K be a closed convex subset of a normed vector space X, (x_n) be a sequence of points in K converging weakly to x. Then $x \in K$.

An existence theorem

Second proof

- Fix $T \in H^{-1}(\Omega)$ and define the 'variational energy':

$$I[v] = \frac{1}{2}B(v,v) - Tv \text{ for } v \in X := H^1_0(\Omega).$$

  The key point of the proof is the fact that: *$u \in X$ solves $Lu = T$ if u is a minimizer or I on X i.e. $I[u] \leq I[v]$ for all $v \in X$.*

- Step 1: Boundedness of minimizing sequence.
  Let $\alpha = \inf_X I \in \mathbb{R} \cup \{-\infty\}$. Note that $I[0] = 0$ and so $\alpha \leq 0$. Pick $u_m \in X$ such that $I[u_m] \to \alpha$. We show that the sequence (u_m) is bounded in $H^1(\Omega)$.
  - ⋆ By the ellipticity and the non-negativity of c, we have

$$B(u_m, u_m) = \int_\Omega [a_{ij}\partial_j u_m \partial_i u_m + cu_m^2]\,dx \geq \lambda \int_\Omega |\nabla u_m|^2\,dx.$$

An existence theorem

Second proof

- Step 1: Boundedness of minimizing sequence (u_m).
  - ⋆ Hence, by Friedrichs' inequality, $B(u_m, u_m) \geq \frac{1}{C}\|u_m\|_X^2$.
  - ⋆ It follows that

$$I[u_m] = \frac{1}{2}B(u_m, u_m) - Tu_m \geq \frac{1}{2C}\|u_m\|_X^2 - \|T\|\|u_m\|_X$$
$$\geq \frac{1}{4C}\|u_m\|_X^2 - C\|T\|^2.$$

  - ⋆ On the other hand, as $I[u_m] \to \alpha \leq 0$, we have $(I[u_m])$ is bounded from above. Therefore (u_m) is bounded in X.
- Step 2: The weak convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ Since $H^1(\Omega)$ is reflexive, the bounded sequence (u_m) has a weakly convergent subsequence.
  - ⋆ We still denote this subsequence (u_m) so that $u_m \rightharpoonup u$ in $H^1(\Omega)$.

An existence theorem

Second proof

- Step 2: The weak convergence of (u_m) along a subsequence to a minimizer of I.
  - $\star$ $u_m \rightharpoonup u$ in H^1.
  - $\star$ As X is weakly closed in H^1 and $(u_m) \in X$, we have that $u \in X$.
  - $\star$ By definition of weak convergence, we have $Tu_m \to Tu$. We claim that

$$\liminf_{m\to\infty} B(u_m, u_m) \geq B(u, u). \qquad (*)$$

  Once this is shown, we have that $I[u] \leq \liminf I[u_m] = \alpha$ and so $I[u] = \alpha$.

An existence theorem

Second proof

- Step 2: The convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ We now prove (*), i.e. $\liminf_{m\to\infty} B(u_m, u_m) \geq B(u, u)$.
  - ⋆ To illustrate the idea, let us consider for now the case $c = 0$ and $a_{ij} = \delta_{ij}$. Then

$$B(u_m, u_m) - B(u, u) = \int_\Omega [|\nabla u_m|^2 - |\nabla u|^2]\,dx$$
$$= \int_\Omega |\nabla(u_m - u)|^2\,dx + 2\int_\Omega \nabla(u_m - u)\cdot\nabla u\,dx.$$

The first term is non-negative. The second term converges to 0 as $\nabla(u_m - u) \rightharpoonup 0$ in L^2. Hence

$$\liminf_{m\to\infty}[B(u_m, u_m) - B(u, u)] = \liminf_{m\to\infty}\int_\Omega |\nabla(u_m - u)|^2\,dx \geq 0.$$

An existence theorem

Second proof

- Step 2: The convergence of (u_m) along a subsequence to a minimizer of I.
  - ⋆ The proof in the general case is similar. We compute

$$B(u_m, u_m) - B(u, u) = \int_\Omega [a_{ij}\partial_i(u_m - u)\partial_j(u_m - u) + c(u_m - u)^2] + \int_\Omega \Big[a_{ij}\partial_i(u_m - u)\partial_j u + a_{ij}\partial_i u \partial_j(u_m - u) + 2c(u_m - u)u\Big]\,dx.$$

Again, the first integral is non-negative while the second and third terms tend to zero. The claim (*) follows, and we conclude Step 2.

An existence theorem

Second proof

- Step 3: We show that u solves $Lu = T$, i.e. $B(u, \varphi) = T\varphi$ for all $\varphi \in X$.
  - ⋆ For $t \in \mathbb{R}$, let $H(t) = I[u + t\varphi]$.
  - ⋆ As shown in Step 2, $I[u] \leq I[u + t\varphi]$ for all t. Hence H has a global minimum at $t = 0$.
  - ⋆ Now note that $H(t)$ is a quadratic polynomial in t:

$$H(t) = \frac{1}{2}B(u + t\varphi, u + t\varphi) - T(u + t\varphi)$$
$$= I[u] + \frac{1}{2}t(B(u, \varphi) + B(\varphi, u) - 2T\varphi) + \frac{1}{2}t^2 B(\varphi, \varphi).$$

  - ⋆ We deduce that

$$0 = H'(0) = \frac{1}{2}(B(u, \varphi) + B(\varphi, u) - 2T\varphi).$$

  - ⋆ Since B is symmetric, we deduce that $B(u, \varphi) = T\varphi$ as wanted.

An existence theorem

Second proof

- Step 4: We prove the uniqueness: If $\bar{u}$ also solves $L\bar{u} = T$, then $\bar{u} = u$.
  - ⋆ It suffices to show that if $Lu = 0$, then $u = 0$.
  - ⋆ $Lu = 0$ means $B(u, \varphi) = 0$ for all $\varphi \in X$. In particular $B(u, u) = 0$.
  - ⋆ But we showed in Step 1 that $B(u, u) \geq \frac{1}{C}\|u\|_X^2$. Therefore $u = 0$.

An example of non-existence and non-uniqueness

We now consider a motivating example for our next discussion:

$$\begin{cases} Lu = -u'' - u = f, \\ u(0) = u(\pi) = 0. \end{cases} \qquad (\heartsuit)$$

- This problem has no uniqueness, as the function $v_0(x) = \sin x$ satisfies $Lv_0 = 0$ and $v_0(0) = v_0(\pi) = 0$.
- Furthermore, if $(\heartsuit)$ is solvable, then upon multiplying with v_0 and integrating we get

$$\int_0^\pi f v_0\,dx = \int_0^\pi [-u'' v_0 - u v_0]\,dx = \int_0^\pi [u' v_0' - u v_0]\,dx$$
$$= \int_0^\pi [-u v_0'' - u v_0]\,dx = 0.$$

Hence, when $\int_0^\pi f v_0\,dx \neq 0$, the problem $(\heartsuit)$ is not solvable.

An example of non-existence and non-uniqueness

- No uniqueness. Solvable only if $\int_0^\pi f v_0 \, dx = 0$.
- Conversely, suppose $\int_0^\pi f v_0 \, dx = 0$. If $f \in L^2(0, \pi)$, we can write $f = \sum_{n=2}^\infty f_n \sin nx$ with $(f_n) \in \ell^2$. Formally expanding $u = \sum_{n=1}^\infty u_n \sin nx$ gives

$$u_1 \text{ is arbitrary and } u_n = \frac{f_n}{n^2 - 1} \text{ for } n \geq 2.$$

An example of non-existence and non-uniqueness

- Let us check that $u_* := \sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx$ belongs to $H_0^1(0, \pi)$ and satisfies $Lu_* = f$ in the weak sense.
  - ⋆ The function $\sin nx \in H_0^1(0, \pi)$ and has norm

$$\|\sin nx\|_{H^1}^2 = \int_0^\pi [n^2 \cos^2 nx + \sin^2 nx]\, dx = \frac{(n^2+1)\pi}{2}.$$

  - ⋆ The system $\{\sin nx\}$ is orthogonal in $H^1(0, \pi)$.
  - ⋆ It follows that

$$\Big\| \sum_{m_1 \leq n \leq m_2} \frac{f_n}{n^2-1} \sin nx \Big\|_{H^1}^2 = \sum_{m_1 \leq n \leq m_2} \frac{f_n^2}{(n^2-1)^2} \frac{(n^2+1)\pi}{2}$$
$$\leq \frac{5\pi}{18} \sum_{m_1 \leq n \leq m_2} f_n^2 \stackrel{m_1, m_2 \to \infty}{\longrightarrow} 0.$$

An example of non-existence and non-uniqueness

- We are checking that $u_* := \sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx \in H_0^1(0,\pi)$ and $Lu_* = f$.
  - ⋆ Therefore, the series $\sum_{n=2}^{\infty} \frac{f_n}{n^2 - 1} \sin nx$ converges in H^1 to $u_* \in H_0^1(0,\pi)$.
  - ⋆ To show that $Lu_* = f$, we consider the truncated series $u_{(N)} = \sum_{n=2}^{N} \frac{f_n}{n^2 - 1} \sin nx$ and $f_{(N)} = \sum_{n=2}^{N} f_n \sin nx$. These are smooth functions and satisfy $Lu_{(N)} = f_{(N)}$. The convergence of $u_{(N)}$ to u_* in H^1 and of $f_{(N)}$ to f in L^2 thus implies that $Lu_* = f$ (check this!).

An example of non-existence and non-uniqueness

$$\begin{cases} Lu = -u'' - u = f, \\ u(0) = u(\pi) = 0. \end{cases} \qquad (\heartsuit)$$

- We conclude that, for given $f \in L^2(0, \pi)$, $(\heartsuit)$ is solvable if and only if $\int_0^\pi f v_0 \, dx = 0$. Furthermore, when that is the case, all solutions are of the form $u(x) = u_*(x) + C \sin x$ for some particular solution u_*.
- Exercise: Check that $u_* \in H^2(0, \pi)$.

An obstruction for existence and uniqueness

We now return to the general setting: $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$ is a bounded linear operator from $H^1(\Omega)$ into $H^{-1}(\Omega)$.

- Uniqueness holds if and only if $L|_{H_0^1(\Omega)}$ is injective.
- Existence holds if and only if $L|_{H_0^1(\Omega)}$ is surjective.
- If $u \in H_0^1(\Omega)$ satisfies $Lu = T$, then for all $\varphi \in H_0^1(\Omega)$, we have

$$T\varphi = B(u,\varphi) = \int_\Omega \Big[a_{ij}\partial_j u \partial_i \varphi + b_i \partial_i u \varphi + cu\varphi\Big]\,dx.$$

If we can integrate by parts once more, we then have

$$T\varphi = \int_\Omega u\Big[-\partial_j(a_{ij}\partial_i\varphi) + \partial_i(b_i\varphi) + c\varphi\Big]\,dx.$$

Hence, if v_0 is such that $-\partial_j(a_{ij}\partial_i v_0) + \partial_i(b_i v_0) + cv_0 = 0$ in Ω, then we must necessarily have $Tv_0 = 0$.

The formal adjoint operator

Definition

Let $Lu = -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu$. The formal adjoint L^* of L is defined as the operator $L^* : H^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$L^* v = -\partial_i(a_{ij}\partial_j v) - \partial_i(b_i v) + cv,$$

$$L^* v(\psi) = \int_\Omega \Big[a_{ij}\partial_j\psi\partial_i v + b_i\partial_i\psi v + c\psi v\Big]\,dx \text{ for } \psi \in H_0^1(\Omega).$$

- The formal adjoint satisfies

$$Lu(v) = B(u,v) = L^* v(u) \text{ for all } u, v \in H_0^1(\Omega).$$

- For $v \in H^1(\Omega)$ and $T \in H^{-1}(\Omega)$, we have $L^* v = T$ if and only if

$$B(\psi, v) = T\psi \text{ for all } \psi \in H_0^1(\Omega).$$

The Fredholm alternative

Theorem (Fredholm alternative)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$.

1. *The boundary value problem*

$$\begin{cases} Lu = f + \partial_i g_i & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases} \qquad \text{(BVP)}$$

   is uniquely solvable for each $f \in L^2(\Omega), g \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$ if and only if $L|_{H^1_0(\Omega)}$ is injective.

2. *The kernels N of $L|_{H^1_0(\Omega)}$ and N^* of $L^*|_{H^1_0(\Omega)}$ are finite dimensional, and their dimensions are equal.*

3. *If N is non-trivial, (BVP) has a solution if and only if $B(u_0, v) = \langle f, v\rangle - \langle g_i, \partial_i v\rangle$ for all $v \in N^*$.*