C3.1 Algebraic Topology

Sheet 2

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Convention: all spaces are topological spaces,
maps of spaces are always continuous.

1) Show that chain homotopy of chain maps $C_* \to \tilde{C}_*$ is an equivalence relation.

2) Show that the relative homology $H_1(\mathbb{R}, \mathbb{Q})$ of the pair $\mathbb{Q} \subseteq \mathbb{R}$ is a free abelian group, and find a basis.
[On Sheet 0 ex.4 you did this by hand, this time use the course!]

3) In the course notes, from a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ we built the "long exact sequence" (LES):

$$\cdots \to H_*(A) \xrightarrow{i_*} H_*(B) \xrightarrow{\pi_*} H_*(C) \xrightarrow{\delta} H_{*-1}(A) \xrightarrow{i_*[-1]} \cdots$$

In the notes we showed exactness at $H_*(C)$ (i.e. $\ker \delta = \operatorname{Im} \pi_*$)
Prove exactness at $H_*(A)$ and $H_*(B)$ in the LES.

4) a) Use the excision theorem to prove that: if each $x_i \in X_i$ has a neighbourhood that can be contracted to x_i, and $\{x_i\} \subseteq X_i$ is a closed set, (i.e. (X_i, x_i) is a "good pair")

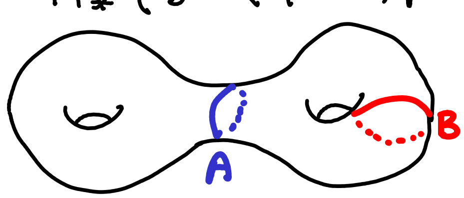
$$\tilde{H}_*(\bigvee_i X_i) \cong \bigoplus_i \tilde{H}_*(X_i)$$

recall the **wedge sum** $\bigvee X_i = \underbrace{\bigsqcup X_i}_{\text{identify all } x_i}$

b) Construct a topological space X such that for all $k \geq 0$,
$$H_k(X) \cong \mathbb{Z}^{n_k} \quad \text{where } n_k \in \mathbb{N} \text{ are arbitrary}$$

c) Construct a connected topological space X with the same homology groups as the torus T^2, which is not homotopy equivalent to T^2. (so in particular not homeomorphic to T^2)

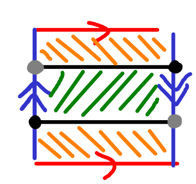
5) a) Compute $H_*(S^n \setminus (k+1) \text{ points})$ and $H_*(\mathbb{R}^2 \setminus k \text{ points})$

b) $\Sigma_2 =$ 

Compute $H_*(\Sigma_2, A)$, $H_*(\Sigma_2, B)$.

c) Using Mayer–Vietoris, calculate

$H_*(S^n)$, $H_*(\text{Klein bottle } K)$ view K = gluing of 2 Möbius bands along a boundary circle



6) Build an explicit homeomorphism

$$D^n/S^{n-1} \cong S^n$$

in a way that preserves the orientations

Hint parametrise points of D^n by $(t\cdot x_1, \dots, t\cdot x_n)$ where $(x_1,\dots,x_n)\in S^{n-1}$ and $t\in[0,1]$

7) If X retracts onto A, prove that $H_*(X) \cong H_*(A) \oplus H_*(X,A)$.

8) a) Viewing paths as singular 1-chains ($\Delta^1 \cong I$), prove that a constant path c is a boundary: $c \in \partial C_2(X)$

b) For paths $f,g: I \to X$ with $f(1)=g(0)$, let $f*g: I \to X$ be the **concatenated path**: $f*g(t) = f(t)$ for $t\in[0,\frac{1}{2}]$, $g(2t-1)$ for $t\in[\frac{1}{2},1]$.

Prove that: $\quad f*g - f - g \in \partial C_2(X)$

c) Let f^{-1} denote the **reversed path**: $f^{-1}(t) = f(t-1)$. Prove

$$f + f^{-1} \in \partial C_2(X)$$

d) If f,g are homotopic paths relative to ∂I, prove

$$f - g \in \partial C_2(X)$$

e) Deduce that $\exists$ group homomorphism (**Hurewicz homomorphism**) $\pi_1(X,x) \to \pi_1^{ab}(X,x) \to H_1(X)$, where π_1^{ab} is the abelianisation.

f) Assume from now on that X is path-connected. Fix $x\in X$. Pick a path $\gamma_y : I \to X$ from x to y, for each $y \in X$, with $\gamma_x \equiv x$. Show $\exists$ hom $H_1(X) \to \pi_1^{ab}(X,x)$ which on chains is the gp. hom: $\varphi: C_1(X) \to \pi_1^{ab}(X,x)$, $\varphi(f: I\to X) = \gamma_{f(0)} * f * \gamma_{f(1)}^{-1}$.

Deduce that **$H_1(X) \cong \pi_1^{ab}(X,x)$** for any path-connected X.

9) Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\setminus 0\}$

$H = \bigcup_{n\in\mathbb{N}\setminus 0} \{\text{circle centre } (\frac{1}{n},0) \text{ and radius } \frac{1}{n}\} \subseteq \mathbb{R}^2$

H

"Hawaiian earring"

$W = \bigvee_{n\in\mathbb{N}\setminus 0} S^1 = \bigsqcup_{n\in\mathbb{N}\setminus 0} S^1 \,/\,$ identify $(-1,0)\in S^1$ in each copy of S^1.

- Show that H, W are not homeomorphic
- Is X/A homeomorphic to H or to W?
- Show that $H_1(X,A) \not\cong \tilde{H}_1(X/A)$ (note $A \subseteq X$ is not a good pair)

(You do not need to fully compute $\tilde{H}_1(X/A)$, that is tricky. Ex. 8 helps).