BO1 History of Mathematics
Lecture XIV
Geometry and number theory

MT 2021 Week 7

Summary

Part 1

- Euclid's *Elements* revisited
- The parallel postulate
- Non-Euclidean geometry

Part 2

- Number theory down the centuries

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

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Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

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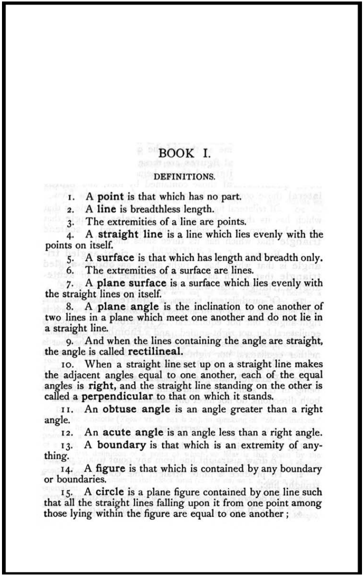
Euclid in English



BOOK I.

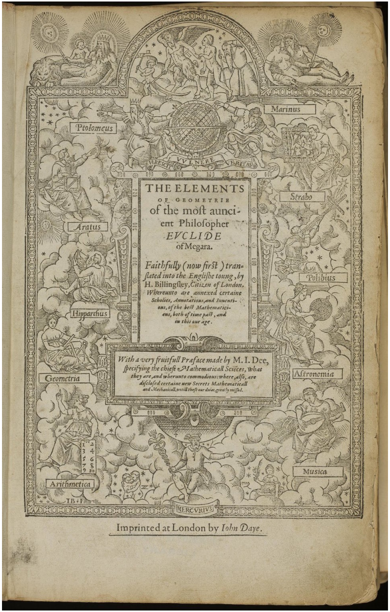
DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal**.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

Canonical English edition by Sir Thomas L. Heath, 1908

See also the Reading Euclid Project

Billingsley's Euclid, 1570



THE ELEMENTS
OF GEOMETRIE
of the moſt aunci-
ent Philoſopher
EVCLIDE
of Megara.

Faithfully (now firſt) tran-
ſlated into the Engliſhe toung, by
H. Billingſley, Citizen of London.
Whereunto are annexed certaine
Scholies, Annotations, and Inuenti-
ons, of the beſt Mathematici-
ens, both of time paſt, and
in this our age.

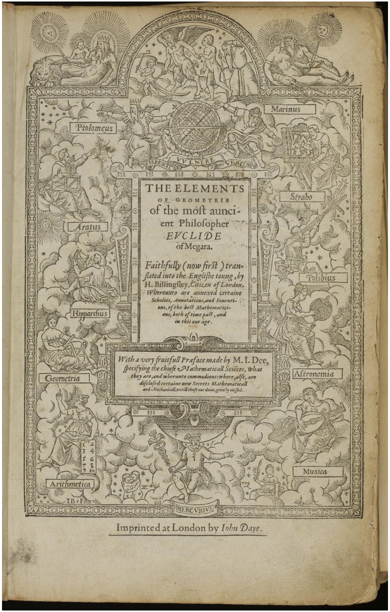
With a very fruitfull Præface made by M. I. Dee,
ſpecifying the chiefe Mathematicall Sciēces, what
they are, and wherunto commodious: where, alſo, are
diſcloſed certaine new Secrets Mathematicall
and Mechanicall, vntill theſe our daies, greatly miſſed.

Imprinted at London by Iohn Daye.

The Elements of Geometrie:

"Faithfully (now first) translated into the Englishe toung" by H. Billingsley, London, 1570

Billingsley's Euclid, 1570



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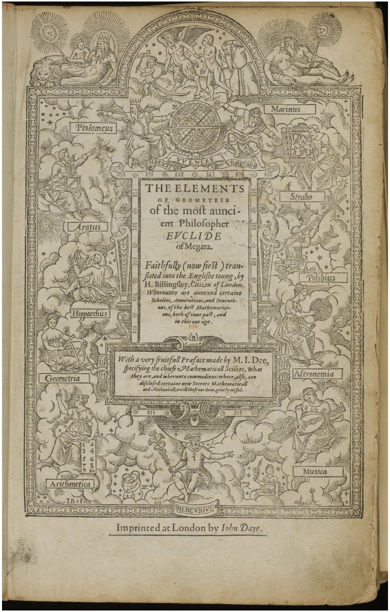

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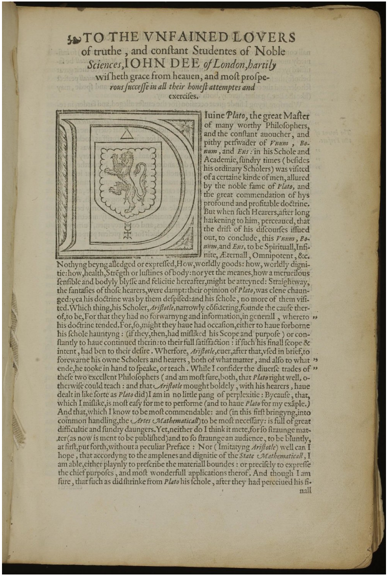
Preface by John Dee

Dee's Preface



TO THE VNFAINED LOVERS of truthe, and conſtant Studentes of Noble Sciences, IOHN DEE *of London, hartily* wiſheth grace from heauen, and moſt proſpe-*rous ſucceſſe in all their honeſt attemptes and* exerciſes.

Diuine *Plato*, the great Maſter of many worthy Philoſophers, and the conſtant auoucher, and pithy perſwader of *Vnum*, *Bonum*, and *Ens*: in his Schole and Academie, ſundry times (beſides his ordinary Scholers) was viſited of a certaine kinde of men, allured by the noble fame of *Plato*, and the great commendation of hys profound and profitable doctrine. But when ſuch Hearers, after long harkening to him, perceaued, that the drift of his diſcourſes iſſued out, to conclude, this *Vnum*, *Bonum*, and *Ens*, to be Spirituall, Infinite, Æternall, Omnipotent, &c. Nothyng beyng alledged or expreſſed, How, worldly goods: how, worldly dignitie: how, health, Strẽgth or luſtines of body: nor yet the meanes, how a merueilous ſenſible and bodyly blyſſe and felicitie hereafter, might be atteyned: Straightway, the fantaſies of thoſe hearers, were dampt: their opinion of *Plato*, was clene chaunged: yea his doctrine was by them deſpiſed: and his ſchole, no more of them viſited. Which thing, his Scholer, *Ariſtotle*, narrowly cõſidering, founde the cauſe therof, to be, For that they had no forwarnyng and information, in generall, whereto his doctrine tended. For, ſo, might they haue had occaſion, either to haue forborne his ſchole hauntyng: (if they, then, had miſliked his Scope and purpoſe) or conſtantly to haue continued therin: to their full ſatiſfaction: if ſuch his finall ſcope & intent, had ben to their deſire. Wherfore, *Ariſtotle*, euer, after that, vſed in brief, to forewarne his owne Scholers and hearers, both of what matter, and alſo to what ende, he tooke in hand to ſpeake, or teach. While I conſider the diuerſe trades of theſe two excellent Philoſophers (and am moſt ſure, both, that *Plato* right well, otherwiſe could teach: and that *Ariſtotle* mought boldely, with his hearers, haue dealt in like ſorte as *Plato* did) I am in no little pang of perplexitie: Bycauſe, that, which I miſlike, is moſt eaſy for me to performe (and to haue *Plato* for my exãple.) And that, which I know to be moſt commendable: and (in this firſt bringyng, into common handling, the *Artes Mathematicall*) to be moſt neceſſary: is full of great difficultie and ſundry daungers. Yet, neither do I think it mete, for ſo ſtraunge matter (as now is ment to be publiſhed) and to ſo ſtraunge an audience, to be bluntly, at firſt, put forth, without a peculiar Preface: Nor (Imitatyng *Ariſtotle*) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall*, I am able, either playnly to preſcribe the materiall boundes: or preciſely to expreſſe the chief purpoſes, and moſt wonderfull applications therof. And though I am ſure, that ſuch as did ſhrinke from *Plato* his ſchole, after they had perceiued his fi-

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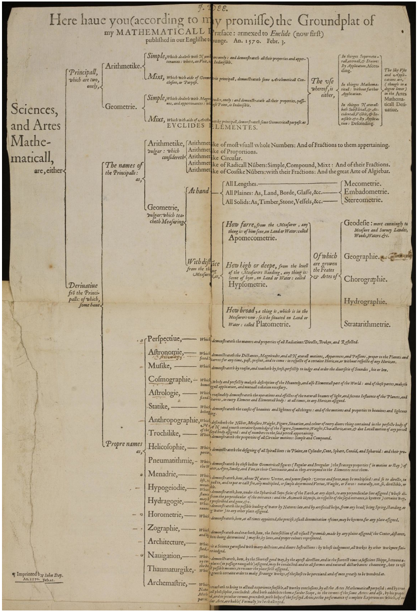
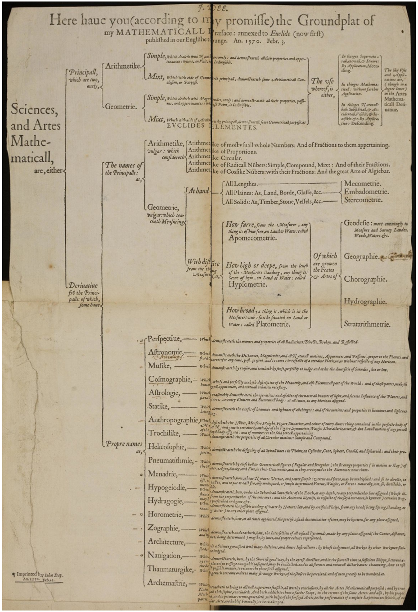
Dee's 'Groundplat'

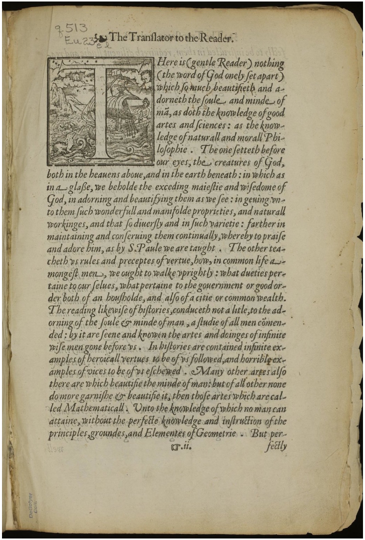


Dee's 'Groundplat'



See: Jennifer M. Rampling, 'The Elizabethan mathematics of everything: John Dee's 'Mathematicall praeface' to Euclid's *Elements*', *BSHM Bulletin: Journal of the British Society for the History of Mathematics* **26**(3) (2011) 135–146

Billingsley's Preface, pp. 1, 3



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There is (gentle Reader) nothing (the word of God onely ſet apart) which ſo much beautifieth and adorneth the ſoule and minde of mã, as doth the knowledge of good artes and ſciences: as the knowledge of naturall and morall Philoſophie. The one ſetteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glaße, we beholde the exceding maieſtie and wiſedome of God, in adorning and beautifying them as we ſee: in geuing vnto them ſuch wonderfull and manifolde proprieties, and naturall workinges, and that ſo diuerſly and in ſuch varietie: farther in maintaining and conſeruing them continually, whereby to praiſe and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongeſt men, we ought to walke vprightly: what dueties pertaine to our ſelues, what pertaine to the gouernment or good order both of an houſholde, and alſo of a citie or common wealth. The reading likewiſe of hiſtories, conduceth not a litle, to the adorning of the ſoule & minde of man, a ſtudie of all men cõmended: by it are ſeene and knowen the artes and doinges of infinite wiſe men gone before vs. In hiſtories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eſchewed. Many other artes alſo there are which beautifie the minde of man: but of all other none do more garniſhe & beautifie it, then thoſe artes which are called Mathematicall. Unto the knowledge of which no man can attaine, without the perfecte knowledge and inſtruction of the principles, groundes, and Elementes of Geometrie. But perfectly

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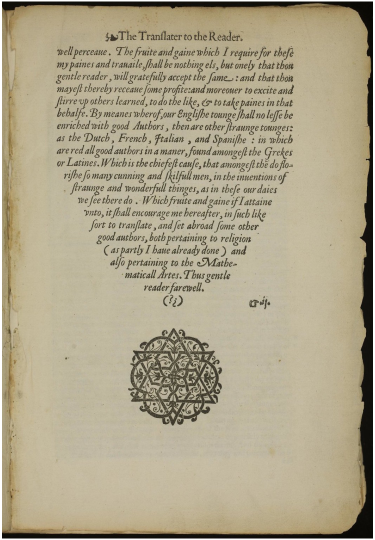

well perceaue. The fruite and gaine which I require for theſe my paines and trauaile, ſhall be nothing els, but onely that thou gentle reader, will gratefully accept the ſame: and that thou mayeſt thereby receaue ſome profite: and moreouer to excite and ſtirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Engliſhe tounge ſhall no leſſe be enriched with good Authors, then are other ſtraunge tounges: as the Dutch, French, Italian, and Spaniſhe: in which are red all good authors in a maner, found amongeſt the Grekes or Latines. Which is the chiefeſt cauſe, that amongeſt thẽ do floriſhe ſo many cunning and ſkilfull men, in the inuentions of ſtraunge and wonderfull thinges, as in theſe our daies we ſee there do. Which fruite and gaine if I attaine vnto, it ſhall encourage me hereafter, in ſuch like ſort to tranſlate, and ſet abroad ſome other good authors, both pertaining to religion (as partly I haue already done) and alſo pertaining to the Mathematicall Artes. Thus gentle reader farewell.

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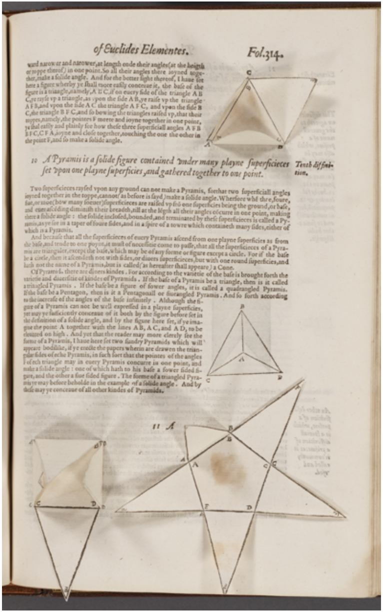
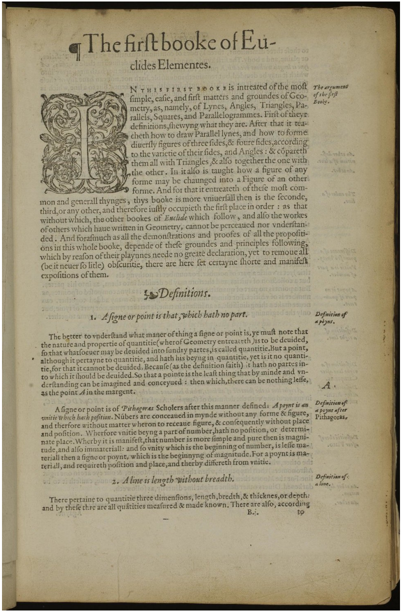
Pop-up Euclid



Book I: definitions



¶ The firſt booke of Euclides Elementes.

IN THIS FIRST BOOKE is intreated of the moſt ſimple, eaſie, and firſt matters and groundes of Geometry, as, namely, of Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. Firſt of theyr definitions, ſhewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuerſly figures of three ſides, & foure ſides, according to the varietie of their ſides, and Angles: & cōpareth them all with Triangles, & alſo together the one with the other. In it alſo is taught how a figure of any forme may be chaunged into a Figure of an other forme. And for that it entreateth of theſe moſt common and generall thynges, thys booke is more vniuerſall then is the ſeconde, third, or any other, and therefore iuſtly occupieth the firſt place in order: as that without which, the other bookes of *Euclide* which follow, and alſo the workes of others which haue written in Geometry, cannot be perceaued nor vnderſtanded. And foraſmuch as all the demonſtrations and proofes of all the propoſitions in this whole booke, depende of theſe groundes and principles following, which by reaſon of their playnnes neede no greate declaration, yet to remoue all (be it neuer ſo litle) obſcuritie, there are here ſet certayne ſhorte and manifeſt expoſitions of them.

The argumentes of the firſt Booke.

Definitions.

1. *A ſigne or point is that, which hath no part.*

Definition of a poynt.

The better to vnderſtand what maner of thing a ſigne or point is, ye muſt note that the nature and propertie of quantitie (wherof Geometry entreateth) is to be deuided, ſo that whatſoeuer may be deuided into ſundry partes, is called quantitie. But a point, although it pertayne to quantitie, and hath his beyng in quantitie, yet is it no quantitie, for that it cannot be deuided. Becauſe (as the definition ſaith) it hath no partes into which it ſhould be deuided. So that a pointe is the leaſt thing that by minde and vnderſtanding can be imagined and conceyued: then which, there can be nothing leſſe, as the point *A* in the margent.

A.

Definition of a poynt after Pithagoras.

A ſigne or point is of *Pithagoras* Scholers after this manner defined: *A poynt is an vnitie which hath poſition.* Nūbers are conceaued in mynde without any forme & figure, and therfore without matter wheron to receaue figure, & conſequently without place and poſition. Wherfore vnitie beyng a part of number, hath no poſition, or determinate place. Wherby it is manifeſt, that number is more ſimple and pure then is magnitude, and alſo immateriall: and ſo vnity which is the beginning of number, is leſſe materiall then a ſigne or poynt, which is the beginnyng of magnitude. For a poynt is materiall, and requireth poſition and place, and therby differeth from vnitie.

2. *A line is length without breadth.*

Definition of a line.

There pertaine to quantitie three dimenſions, length, bredth, & thicknes, or depth: and by theſe thre are all quātities meaſured & made known. There are alſo, according

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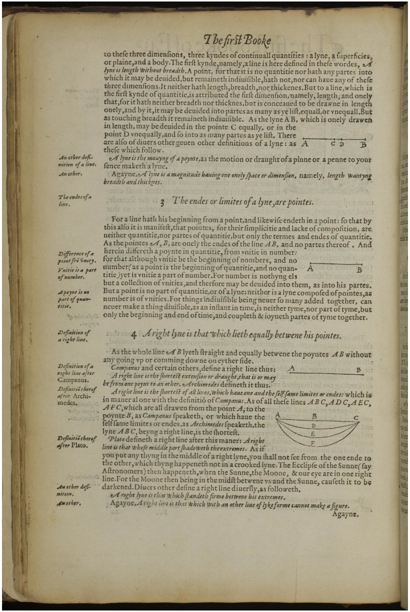


to theſe three dimenſions, three kyndes of continuall quantities: a lyne, a ſuperficies, or plaine, and a body. The firſt kynde, namely, a line is here defined in theſe wordes, *A lyne is length without breadth.* A point, for that it is no quantitie nor hath any partes into which it may be deuided, but remaineth indiuiſible, hath not, nor can haue any of theſe three dimenſions. It neither hath length, breadth, nor thickenes. But to a line, which is the firſt kynde of quantitie, is attributed the firſt dimenſion, namely, length, and onely that, for it hath neither breadth nor thicknes, but is conceaued to be drawne in length onely, and by it, it may be deuided into partes as many as ye liſt, equall, or vnequall. But as touching breadth it remaineth indiuiſible. As the lyne A B, which is onely drawen in length, may be deuided in the pointe C equally, or in the point D vnequally, and ſo into as many partes as ye liſt. There are alſo of diuers other geuen other definitions of a lyne: as theſe which follow.

An other definition of a line.

A lyne is the mouyng of a poynte, as the motion or draught of a pinne or a penne to your ſence maketh a lyne.

An other.

Agayne, *A lyne is a magnitude hauing one onely ſpace or dimenſion,* namely, *length wantyng breadth and thicknes.*

The endes of a line.

3 *The endes or limites of a lyne, are pointes.*

For a line hath his beginning from a point, and likewiſe endeth in a point: ſo that by this alſo it is manifeſt, that pointes, for their ſimplicitie and lacke of compoſition, are neither quantitie, nor partes of quantitie, but only the termes and endes of quantitie. As the pointes *A*, *B*, are onely the endes of the line *A B*, and no partes thereof. And herein differeth a poynte in quantitie, from vnitie in number: for that although vnitie be the beginning of nombers, and no number (as a point is the beginning of quantitie, and no quantitie) yet is vnitie a part of number. For number is nothyng els but a collection of vnities, and therfore may be deuided into them, as into his partes. But a point is no part of quantitie, or of a lyne: neither is a lyne compoſed of pointes, as number is of vnities. For things indiuiſible being neuer ſo many added together, can neuer make a thing diuiſible, as an inſtant in time, is neither tyme, nor part of tyme, but only the beginning and end of time, and coupleth & ioyneth partes of tyme together.

Difference of a point frō vnity.

Vnitie is a part of number.

A poynt is no part of quantitie.

Definition of a right line.

4 *A right lyne is that which lieth equally betwene his pointes.*

As the whole line *A B* lyeth ſtraight and equally betwene the poyntes *A B* without any going vp or comming downe on eyther ſide.

Definition of a right line after Campanus.

Campanus and certain others, define a right line thus: *A right line is the ſhorteſt extenſion or draught, that is or may be from one poynt to an other.* *Archimedes* defineth it thus.

Definitiō therof after Archimedes.

A right line is the ſhorteſt of all lines, which haue one and the ſelf ſame limites or endes: which is in maner al one with the definitiō of *Campanus*. As of all theſe lines *A B C*, *A D C*, *A E C*, *A F C*, which are all drawen from the point *A*, to the poynte *B*, as *Campanus* ſpeaketh, or which haue the ſelf ſame limites or endes, as *Archimedes* ſpeaketh, the lyne *A B C*, beyng a right line, is the ſhorteſt.

Definitiō therof after Plato.

Plato defineth a right line after this maner: *A right line is that whoſe middle part ſhadoweth the extremes.* As if you put any thyng in the middle of a right lyne, you ſhall not ſee from the one ende to the other, which thyng happeneth not in a crooked lyne. The Ecclipſe of the Sunne (ſay Aſtronomers) then happeneth, when the Sunne, the Moone, & our eye are in one right line. For the Moone then being in the midſt betwene vs and the Sunne, cauſeth it to be darkened. Diuers other define a right line diuerſly, as followeth.

An other definition.

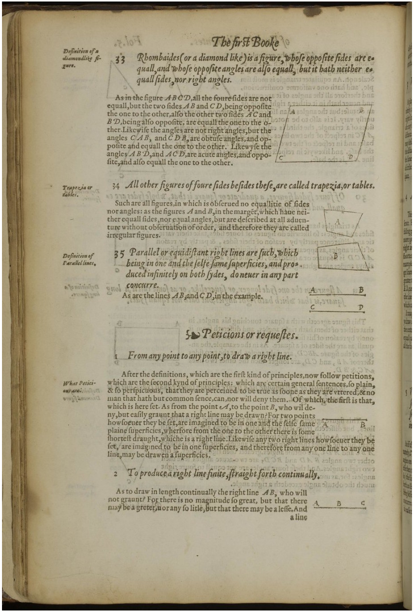
A right lyne is that which ſtandeth firme betwene his extremes.

Another.

Agayne, *A right line is that which with an other line of lyke forme cannot make a figure.*

Agayne,

Book I: postulates



The firſt Booke

Definition of a diamondlike figure.

33 *Rhombaides (or a diamond like) is a figure, whoſe oppoſite ſides are equall, and whoſe oppoſite angles are alſo equall, but it hath neither equall ſides, nor right angles.*

As in the figure *ABCD*, all the foure ſides are not equall, but the two ſides *AB* and *CD*, being oppoſite the one to the other, alſo the other two ſides *AC* and *BD*, being alſo oppoſite, are equall the one to the other. Likewiſe the angles are not right angles, but the angles *CAB*, and *CDB*, are obtuſe angles, and oppoſite and equall the one to the other. Likewyſe the angles *ABD*, and *ACD*, are acute angles, and oppoſite, and alſo equall the one to the other.

Trapezia or tables.

34 *All other figures of foure ſides beſides theſe, are called trapezia, or tables.*

Such are all figures, in which is obſerued no equallitie of ſides nor angles: as the figures *A* and *B*, in the margēt, which haue neither equall ſides, nor equal angles, but are deſcribed at all aduenture without obſeruation of order, and therefore they are called irregular figures.

Definition of Parallel lines.

35 *Parallel or equidiſtant right lines are ſuch, which being in one and the ſelfe ſame ſuperficies, and produced infinitely on both ſydes, do neuer in any part concurre.*

As are the lines *AB*, and *CD*, in the example.

Peticions or requeſtes.

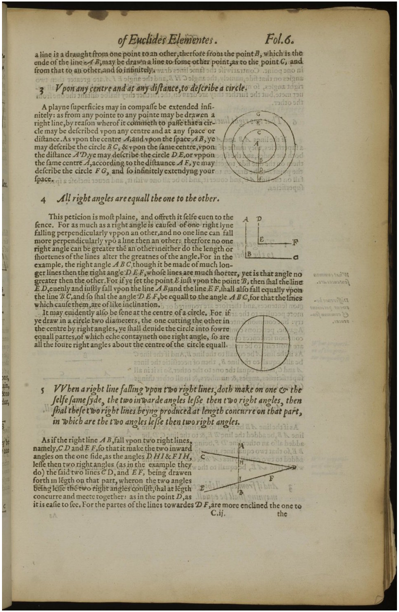
1 *From any point to any point, to draw a right line.*

What Peticions are.

After the definitions, which are the firſt kind of principles, now follow petitions, which are the ſecond kynd of principles: which are certain general ſentences, ſo plain, & ſo perſpicuous, that they are perceiued to be true as ſoone as they are vttered, & no man that hath but common ſence, can, nor will deny them. Of which, the firſt is that, which is here ſet. As from the point *A*, to the point *B*, who wil deny, but eaſily graunt that a right line may be drawn? For two points howſoeuer they be ſet, are imagined to be in one and the ſelfe ſame plaine ſuperficies, wherfore from the one to the other there is ſome ſhorteſt draught, whiche is a right line. Likewiſe any two right lines howſoeuer they be ſet, are imagined to be in one ſuperficies, and therefore from any one line to any one line, may be drawen a ſuperficies.

2 *To produce a right line finite, ſtraight forth continually.*

As to draw in length continually the right line *AB*, who will not graunt? For there is no magnitude ſo great, but that there may be a greter, nor any ſo litle, but that there may be a leſſe. And a line



of Euclides Elementes. Fol. 6.

a line is a draught from one point to an other, therfore from the point *B*, which is the ende of the line *AB*, may be drawn a line to ſome other point, as to the point *C*, and from that to an other, and ſo infinitely.

3 *Vpon any centre and at any diſtance, to deſcribe a circle.*

A playne ſuperficies may in compaſſe be extended infinitely: as from any pointe to any pointe may be drawen a right line, by reaſon wherof it commeth to paſſe that a circle may be deſcribed vpon any centre and at any ſpace or diſtance. As vpon the centre *A*, and vpon the ſpace *AB*, ye may deſcribe the circle *BC*, & vpon the ſame centre, vpon the diſtance *AD*, ye may deſcribe the circle *DE*, or vppon the ſame centre *A*, according to the diſtaunce *AF*, ye may deſcribe the circle *FG*, and ſo infinitely extendyng your ſpace.

4 *All right angles are equall the one to the other.*

This peticion is moſt plaine, and offreth it ſelfe euen to the ſence. For as much as a right angle is cauſed of one right lyne falling perpendicularly vppon an other, and no one line can fall more perpendicularly vpō a line then an other: therfore no one right angle can be greater thē an other: neither do the length or ſhortenes of the lines alter the greatnes of the angle. For in the example, the right angle *ABC*, though it be made of much longer lines then the right angle *DEF*, whoſe lines are much ſhorter, yet is that angle no greater then the other. For if ye ſet the point *E* iuſt vpon the point *B*, then ſhal the line *ED*, euenly and iuſtly fall vpon the line *AB*, and the line *EF*, ſhall alſo fall equally vpon the line *BC*, and ſo ſhal the angle *DEF*, be equall to the angle *ABC*, for that the lines which cauſe them, are of like inclination.

It may euidently alſo be ſene at the centre of a circle. For if ye draw in a circle two diameters, the one cutting the other in the centre by right angles, ye ſhall deuide the circle into fowre equall partes, of which eche contayneth one right angle, ſo are all the foure right angles about the centre of the circle equall.

5 *When a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.*

As if the right line *AB*, fall vpon two right lines, namely, *CD* and *EF*, ſo that it make the two inward angles on the one ſide, as the angles *DHI* & *FIH*, leſſe then two right angles (as in the example they do) the ſaid two lines *CD*, and *EF*, being drawen forth in lēgth on that part, wheron the two angles being leſſe thē two right angles conſiſt, ſhal at lēgth concurre and meete together: as in the point *D*, as it is eaſie to ſee. For the partes of the lines towardes *DF*, are more enclined the one to the

C.ij.

Postulate 5

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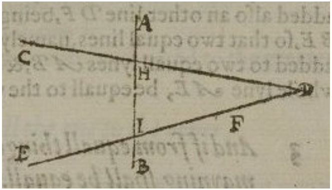

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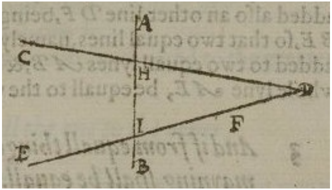

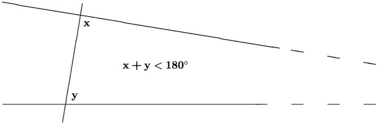

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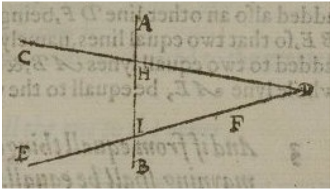

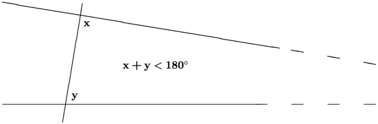

Equivalent formulation (Proclus, 5th century; John Playfair, 1795): given a straight line L and a point P not on L there is one and only one straight line through P that is parallel to L.

Classical disquiet about the fifth postulate

Original to Euclid?

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See Heath, pp. 202–220

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Al-Tusi's thoughts found their way into Europe via the writings (1298) of his son Sadr al-Tusi

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After reading al-Tusi, John Wallis showed that the parallel postulate is equivalent to the following:

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Similar results derived by Johann Heinrich Lambert (1728–1777) in his *Theorie der Parallellinien* (1766)

Non-Euclidean geometries

Consistent non-Euclidean geometry probably first constructed (tentatively) by Gauss, c. 1817–1830, but remained unpublished

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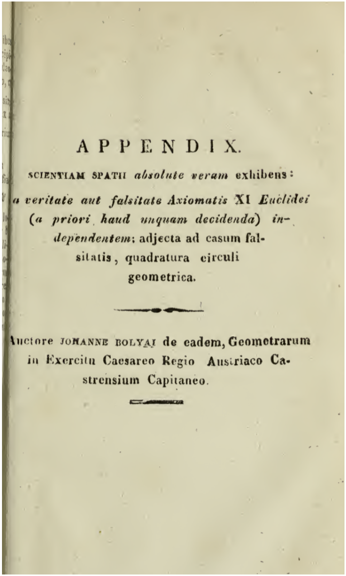
Pursued (against paternal advice) and solved by János Bolyai (1802–1860): "I have created a new and different world out of nothing" (1823)

Bolyai's geometry

> *He [Farkas Bolyai] gave me the opinion that if I was really successful, I should quickly make a public announcement and that for two reasons. First because the idea might easily pass to someone else who would then publish it; second because, and this is another truth, several things ripen at the same time and appear in different places in the manner of violets coming to light in early spring, and since all scientific striving is like a great war in which one does not know when peace will come, one must win, if possible; for here preeminence comes to him who is first.*

Janos Bolyai, 1823 - as quoted in Barrow-Green, Gray & Wilson, Volume 2.

Bolyai's geometry


APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:
a veritate aut falsitate Axiomatis XI *Euclidei*
(*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

Published as appendix 'The science absolute of space: independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)' to father's textbook *Tentamen iuventutem studiosam in elementa matheosos introducendi* (1832)

English translation by George Bruce Halstead (1896)

Meanwhile in Russia...

Non-Euclidean geometry developed independently by Nikolai Ivanovich Lobachevskii [Николай Иванович Лобачевский] (1792–1856) using the negation of Playfair's version of the postulate

Lobachevskii's works

Geometrische Untersuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatschewsky,

Kaiserl. russ. wirkl. Staatsrathe und ord. Prof. der Mathematik
bei der Universität Kasan.

Berlin. 1840.

In der G. Fincke'schen Buchhandlung

Complicated story of dissemination...

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Other works in Russian, French and German, including *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840), *Pangéométrie* (1855)

Acceptance and impact of non-Euclidean geometries

Slow to gain acceptance due to

- obscurity of publications

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- overturned old ideas of mathematical certainty
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- helped drive the late 19th-century move towards axiomatisation

Part 2: Early Number Theory

Euclid on numbers

Euclid on numbers (positive integers)

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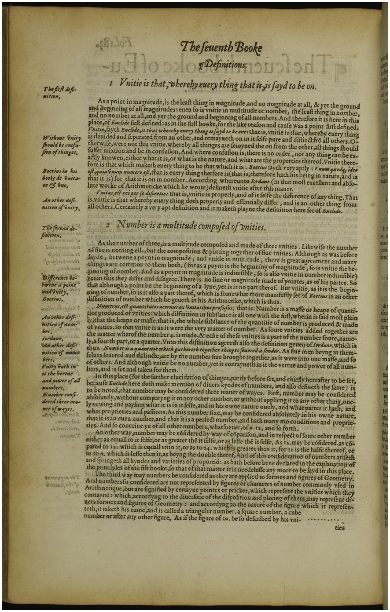


The ſeuenth Booke

¶Definitions.

1 *Vnitie is that, whereby euery thing that is, is ſayd to be on.*

The firſt definition.

As a point in magnitude, is the leaſt thing in magnitude, and no magnitude at all, & yet the ground and beginning of all magnitudes: euen ſo is vnitie in multitude or nomber, the leaſt thing in nomber, and no nomber at all, and yet the ground and beginning of all nombers. And therefore it is here in this place, of *Euclide* firſt defined: as in the firſt booke, for the like reaſon and cauſe was a point firſt defined. *Vnitie*, ſayth *Euclide, is that whereby euery thing is ſayd to be one:* that is, vnitie is that, whereby euery thing is deuided and ſeperated from an other, and remayneth on in it ſelfe pure and diſtinct frõ all others. Otherwiſe, were not this vnitie, whereby all thinges are ſeioyned the on from the other, all things ſhould ſuffer mixtion and be in confuſion. And where confuſion is, there is no order, nor any thing can be exactly knowen, either what it is, or what is the nature, and what are the properties thereof. Vnitie therefore is that which maketh euery thing to be that which it is. *Boetius* ſayth very aptly: *Vnum quodq; ideo eſt, quia Vnum numero eſt*, that is euery thing therfore is (that is, therefore hath his being in nature, and is that it is) for that it is on in nomber. According whereunto *Iordane* (in that moſt excellent and abſolute worke of Arithmeticke which he wrote) defineth vnitie after this maner.

Without Vnity ſhould be confuſion of thinges.

Boetius in his booke de Vnitate & Vno.

Vnitas, eſt rei per ſe diſcretio: that is, vnitie is properly, and of it ſelfe the difference of any thing. That is, vnitie is that whereby euery thing doth properly and eſſentially differ, and is an other thing from all others. Certainely a very apt definition and it maketh playne the definition here ſet of *Euclide*.

An other definition of Vnity.

2 *Number is a multitude compoſed of vnities.*

The ſecond definition.

As the number of three, is a multitude compoſed and made of three vnities. Likewiſe the number of fiue is nothing ells, but the compoſition & putting together of fiue vnities. Although as was before ſayde, betwene a poynt in magnitude, and vnitie in multitude, there is great agreement and many thinges are common to them both, (for as a poynt is the beginning of magnitude, ſo is vnitie the beginning of nomber. And as a poynt in magnitude is indiuiſible, ſo is alſo vnitie in number indiuiſible) yet in this they differ and diſagree. There is no line or magnitude made of pointes, as of his partes. So that although a point be the beginning of a lyne, yet is it no part therof. But vnitie, as it is the beginning of number, ſo is it alſo a part therof, which is ſomewhat more manifeſtly ſet of *Boetius* in an other diffinition of number which he geueth in his Arithmetike, which is thus.

Difference betwene a point and Vnity. Boetius.

Numerus, eſt quantitatis aceruus ex Vnitatibus profuſus, that is. Number is a maſſe or heape of quantities produced of vnities: which diffinition in ſubſtance is all one with the firſt, wherin is ſaid moſt plainly, that the heape or maſſe, that is, the whole ſubſtance of the quantitie of number is produced & made of vnities. So that vnitie is as it were the very matter of number. As foure vnities added together are the matter wherof the number 4. is made, & eche of theſe vnities is a part of the number foure, namely, a fourth part, or a quarter. Vnto this diffinition agreeth alſo the definition geuen of *Iordane*, which is thus. *Number is a quantitie which gathereth together thinges ſeuered a ſonder*. As fiue men beyng in themſelues ſeuered and diſtincte, are by the number fiue brought together, as it were into one maſſe, and ſo of others. And although vnitie be no number, yet it contayneth in it the vertue and power of all numbers, and is ſet and taken for them.

An other definition of number. Iordane. An other definition of numbers. Vnity hath in it the Vertue and power of all numbers.

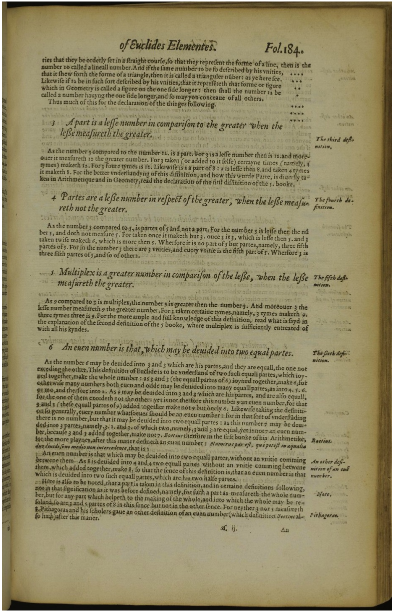
In this place (for the farther elucidation of thinges, partly before ſet, and chiefly hereafter to be ſet, becauſe *Euclide* here doth make mention of diuers kyndes of numbers, and alſo defineth the ſame) is to be noted, that number may be conſidered three maner of wayes. Firſt, number may be conſidered abſolutely, without comparyng it to any other number, or without applieng it to any other thing, onely vewing and payſing what it is in it ſelfe, and in his owne nature onely, and what partes it hath, and what proprieties and paſſions. As this number ſixe, may be conſidered abſolutely in his owne nature, that it is an euen number, and that it is a perfect number, and hath many mo conditions and proprieties. And ſo conceiue ye of all other numbers, whatſoeuer, of 9. 12. and ſo forth.

Number conſidered three maner of wayes.

An other way, number may be cõſidered by way of cõpariſon, and in reſpect of ſome other number either as equall to it ſelfe, or as greater thẽ it ſelfe, or as leſſe thẽ it ſelfe. As 12. may be cõſidered, as cõpared to 12. which is equall vnto it, or as to 24. which is greater then it, for 12 is the halfe thereof, or as to 6. which is leſſe then it, as beyng the double therof. And of this conſideration of numbers ariſeth and ſpringeth all kyndes and varieties of proportiõ: as hath before bene declared in the explanation of the principles of the fift booke, ſo that of that matter it is needeleſſe any more to be ſayd in this place.

The third way may numbers be conſidered as they are applied to formes and figures of Geometry. And numbers ſo conſidered are not repreſented by figures or characters of number commonly vſed in Arithmetique, but are ſignified by certayne pointes or prickes, which repreſent the vnities which they contayne: which, accordyng to the diuerſitie of the diſpoſition and placing of them, may repreſent diuers formes and figures of Geometry: and accordyng to the nature of the figure which it repreſenteth, it taketh his name, and is called a trianguler number, a ſquare number, a cube number or after any other figure. As if the figure of 10. be ſo deſcribed by his vni- ....

ties



of Euclides Elementes. *Fol. 184.*

ties that they be orderly ſet in a ſtraight courſe, ſo that they repreſent the forme of a line, then is the number 10 called a lineall number. And if the ſame number 10 be ſo deſcribed by his vnities, that it ſhew forth the forme of a triangle, then it is called a trianguler nũber: as ye here ſee. Likewiſe if 12 be in ſuch ſort deſcribed by his vnities, that it repreſẽteth that forme or figure which in Geometry is called a figure on the one ſide longer: then ſhall the number 12 be called a number hauyng the one ſide longer, and ſo may you conceaue of all others.

Thus much of this for the declaration of the thinges following.

3 *A part is a leſſe number in compariſon to the greater when the leſſe meaſureth the greater.*

The third definition.

As the number 3 compared to the number 12. is a part. For 3 is a leſſe number then is 12. and moreouer it meaſureth 12 the greater number. For 3 taken (or added to it ſelfe) certayne times (namely, 4 tymes) maketh 12. For 3 foure tymes is 12. Likewiſe is 2 a part of 8: 2 is leſſe then 8, and taken 4 tymes it maketh 8. For the better vnderſtandyng of this diffinition, and how this worde Parte, is diuerſly taken in Arithmetique and in Geomety, read the declaration of the firſt diffinition of the 5. booke.

4 *Partes are a leſſe number in reſpect of the greater, when the leſſe meaſureth not the greater.*

The fourth definition.

As the number 3 compared to 5, is partes of 5 and not a part. For the number 3 is leſſe then the nũber 5, and doth not meaſure 5. For taken once it maketh but 3. once 3 is 3, which is leſſe then 5. and 3 taken twiſe maketh 6, which is more then 5. Wherfore it is no part of 5 but partes, namely, three fifth partes of 5. For in the number 3 there are 3 vnities, and euery vnitie is the fifth part of 5. Wherfore 3 is three fifth partes of 5, and ſo of others.

5 *Multiplex is a greater number in compariſon of the leſſe, when the leſſe meaſureth the greater.*

The fifth definition.

As 9 compared to 3 is multiplex, the number 9 is greater then the number 3. And moreouer 3 the leſſe number meaſureth 9 the greater number. For 3 taken certaine tymes, namely, 3 tymes maketh 9. three tymes three is 9. For the more ample and full knowledge of this definition, read what is ſayd in the explanation of the ſecond definition of the 5 booke, where multiplex is ſufficiently entreated of with all his kyndes.

6 *An euen number is that, which may be deuided into two equal partes.*

The ſixth definition.

As the number 6 may be deuided into 3 and 3 which are his partes, and they are equall, the one not exceding the other. This definition of Euclide is to be vnderſtand of two ſuch equall partes, which ioyned together, make the whole number: as 3 and 3 (the equall partes of 6) ioyned together, make 6, for otherwiſe many numbers both euen and odde may be deuided into many equall partes, as into 4. 5. 6. or mo, and therfore into 2. As 9 may be deuided into 3 and 3 which are his partes, and are alſo equall, for the one of them excedeth not the other: yet is not therfore this number 9 an euen number, for that 3 and 3 (theſe equall partes of 9) added together make not 9 but onely 6. Likewiſe taking the definition ſo generally, euery number whatſoeuer ſhould be an euen number: for in that ſort of vnderſtãding there is no number, but that it may be deuided into two equall partes: as this number 7 may be deuided into 3 partes, namely, 3. 1. and 3. of which two, namely, 3 and 3 are equal, yet is not 7 an euen number, becauſe 3 and 3 added together, make not 7. *Boetius* therfore in the firſt booke of his Arithmetike, for the more playnes, after this maner defineth an euen number: *Numerus par eſt, qui poteſt in aequalia duo diuidi, Vno medio non intercedente*, that is:

Boetius.

An euen number is that which may be deuided into two equall partes, without an vnitie comming betwene them. As 8 is deuided into 4 and 4 two equall partes without an vnitie comming betwene them, which added together, make 8, ſo that the ſence of this definition is, that an euen number is that which is deuided into two ſuch equall partes, which are his two halfe partes.

An other definition of an euẽ number.

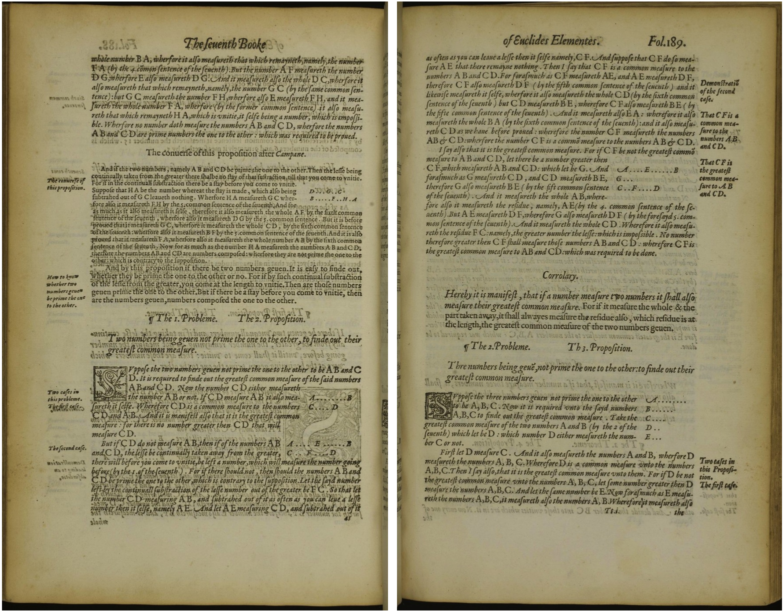
Here is alſo to be noted, that a part is taken in this definition, and in certaine definitions following, not in that ſignification as it was before defined, namely, for ſuch a part as meaſureth the whole number, but for any part which helpeth to the making of the whole, and into which the whole may be reſolued, ſo are 3 and 5 partes of 8 in this ſence but not in the other ſence. For neyther 3 nor 5 meaſureth 8. Pithagoras and his ſcholers gaue an other definition of an euen number (which definition *Boetius* alſo hath) after this maner.

Note.

Pithagoras.

ſ. ij. An

The Euclidean algorithm (Proposition VII.2)



whole number B A, wherfore it alſo meaſureth that which remayneth, namely, the number F A (by the 4. cōmon ſentence of the ſeuenth). But the number A F meaſureth the number D G, wherfore E alſo meaſureth D G. And it meaſureth alſo the whole D C, wherfore it alſo meaſureth that which remayneth, namely, the number G C (by the ſame common ſentence): but G C meaſureth the number F H, wherfore alſo E meaſureth F H, and it meaſureth the whole number F A, wherfore (by the former common ſentence) it alſo meaſureth that which remayneth H A, which is vnitie, it ſelfe being a number, which is impoſſible. Wherfore no number doth meaſure the numbers A B and C D, wherfore the numbers A B and C D are prime numbers the one to the other: which was required to be proued.

The conuerſe of this propoſition after *Campane*.

The conuerſe of this propoſition.

And if the two numbers, namely A B and C D be prime the one to the other. Then the leſſe being continually taken from the greater there ſhalbe no ſtay of that ſuſtraction, till that you come to vnitie. For if in the continuall ſubſtraction there be a ſtay before you come to vnitie. Suppoſe that H A be the number whereat the ſtay is made, which alſo being ſubtrahed out of G C leaueth nothing. Wherfore H A meaſureth G C wherfore alſo it meaſureth F H by the 5. common ſentence of the ſeuenth, And for as much as it alſo meaſureth it ſelfe, therefore it alſo meaſureth the whole A F by the ſixth common ſentence of the ſeuenth, wherfore alſo it meaſureth D G by the 5. common ſentence. But it is before proued that it meaſureth G C, wherfore it meaſureth the whole C D, by the ſixth common ſentence of the ſeuenth: wherfore alſo it meaſureth B F by the 5. common ſentence of the ſeuenth. And it is alſo proued that it meaſureth F A, wherfore alſo it meaſureth the whole number A B by the ſixth common ſentence of the ſeuenth, Now for as much as the number H A meaſureth the numbers A B and C D, therfore the numbers AB and CD are numbers compoſed: wherfore they are not prime the one to the other: which is contrary to the ſuppoſition.

D . . . G . . C
B . . . . . F . . H . A

How to know whether two numbers geuen be prime the one to the other.

And by this propoſition if there be two numbers geuen. It is eaſy to finde out, whether they be prime the one to the other or no. For if by ſuch continual ſubſtraction of the leſſe from the greater, you come at the length to vnitie. Then are thoſe numbers geuen prime the one to the other. But if there be a ſtay before you come to vnitie, then are the numbers geuen, numbers compoſed the one to the other.

¶ *The 1. Probleme.* *The 2. Propoſition.*

Two numbers being geuen not prime the one to the other, to finde out their greateſt common meaſure.

Two caſes in this probleme. The firſt caſe.

Suppoſe the two numbers geuen not prime the one to the other to be A B *and* C D. *It is required to finde out the greateſt common meaſure of the ſaid numbers* A B *and* C D. *Now the number* C D *either meaſureth the number* A B *or not. If* C D *meaſure* A B *it alſo meaſureth it ſelfe. Wherefore* C D *is a common meaſure to the numbers* C D *and* A B. *And it is manifeſt alſo that it is the greateſt common meaſure: for there is no number greater then* C D *that will meaſure* C D.

A . . . . B
C . . . . D

The ſecond caſe.

But if C D *do not meaſure* A B, *then if of the numbers* A B *and* C D, *the leſſe be continually taken away from the greater, there will before you come to vnitie, be left a number, which will meaſure the number going before (by the 1. of the ſeuenth). For if there ſhould not, then ſhould the numbers* A B *and* C D *be prime the one to the other, which is contrary to the ſuppoſition. Let the ſayd number left by the continuall ſubſtraction of the leſſe number out of the greater be* F C. *So that let the number* C D *meaſuring* A B, *and ſubtrahed out of it as often as you can leue a leſſe number then it ſelfe, namely* A E. *And let* A E *meaſuring* C D, *and ſubtrahed out of it*

A . . . . E . . . . . . B
C . . F . . . . D

45

as often as you can leaue a leſſe then it ſelfe namely, C F. *And ſuppoſe that* C F *do ſo meaſure* A E *that there remayne nothing. Then I ſay that* C F *is a common meaſure to the numbers* A B *and* C D. *For foraſmuch as* C F *meaſureth* AE, *and* A E *meaſureth* D F, *therefore* C F *alſo meaſureth* D F *(by the fifth common ſentence of the ſeuenth) and it likewiſe meaſureth it ſelfe, wherfore it alſo meaſureth the whole* C D *(by the ſixth common ſentence of the ſeuenth) but* C D *meaſureth* B E, *wherefore* C F *alſo meaſureth* B E *(by the fifte common ſentence of the ſeuenth). And it meaſureth alſo* E A: *wherefore it alſo meaſureth the whole* B A *(by the ſixth common ſentence of the ſeuenth): and it alſo meaſureth* C D *as we haue before proued: wherefore the number* C F *meaſureth the numbers* AB & C D: *wherfore the number* C F *is a commō meaſure to the numbers* A B & C D.

Demonſtratiō of the ſecond caſe.

That C F is a common meaſure to the numbers A B and C D.

I ſay alſo that it is the greateſt common meaſure. For if C F *be not the greateſt commō meaſure to* A B *and* C D, *let there be a number greater then* C F, *which meaſureth* A B *and* C D: *which let be* G. *And foraſmuch as* G *meaſureth* C D, *and* C D *meaſureth* B E, *therefore* G *alſo meaſureth* B E *(by the fift common ſentence of the ſeuenth). And it meaſureth the whole* A B, *wherefore alſo it meaſureth the reſidue, namely,* A E *(by the 4. common ſentence of the ſeuenth). But* A E *meaſureth* D F, *wherefore* G *alſo meaſureth* D F *(by the foreſayd 5. common ſentence of the ſeuenth). And it meaſureth the whole* C D. *Wherefore it alſo meaſureth the reſidue* F C: *namely, the greater number the leſſe: which is impoſſible. No number therefore greater then* C F *ſhall meaſure thoſe numbers* A B *and* C D: *wherefore* C F *is the greateſt common meaſure to* AB *and* CD: *which was required to be done.*

A . . . . E . . . . . . B
G . . .
C . . F . . . . D

That C F is the greateſt common meaſure to A B and C D.

Corrolary.

Hereby it is manifeſt, that if a number meaſure two numbers it ſhall alſo meaſure their greateſt common meaſure. For if it meaſure the whole & the part taken away, it ſhall alwayes meaſure the reſidue alſo, which reſidue is at the length, the greateſt common meaſure of the two numbers geuen.

¶ *The 2. Probleme.* *Th 3. Propoſition.*

Thre numbers being geuē, not prime the one to the other: to finde out their greateſt common meaſure.

Suppoſe the three numbers geuen not prime the one to the other to be A, B, C. *Now it is required vnto the ſayd numbers* A, B, C *to finde out the greateſt common meaſure. Take the greateſt common meaſure of the two numbers* A *and* B *(by the 2 of the ſeuenth) which let be* D: *which number* D *either meaſureth the number* C *or not.*

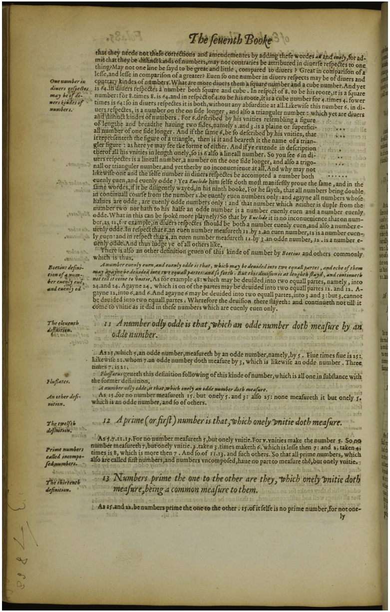
A . . . .
B . . . . . .
C . . . .
D . .
E . . .

Two caſes in this Propoſition. The firſt caſe.

Firſt let D *meaſure* C. *And it alſo meaſureth the numbers* A *and* B, *wherfore* D *meaſureth the numbers* A, B, C. *Wherefore* D *is a common meaſure vnto the numbers* A, B, C. *Then I ſay alſo, that it is the greateſt common meaſure vnto them. For if* D *be not the greateſt common meaſure vnto the numbers* A, B, C, *let ſome number greater then* D *meaſure the numbers* A, B, C. *And let the ſame number be* E. *Now foraſmuch as* E *meaſureth the numbers* A, B, C, *it meaſureth alſo the numbers* A, B. *Wherefore it meaſureth alſo*

Tt.i. the

Euclid on prime numbers



that they neede not theſe corrections and amendementes by adding theſe wordes *all* and *onely*, for admit that they be diſtinct kinds of numbers, may not contraries be attributed in diuerſe reſpectes to one thing? May not one line be ſayd to be great and little, compared to diuers? Great in compariſon of a leſſe, and leſſe in compariſon of a greater? Euen ſo one number in diuers reſpects may be of diuers and contrary kindes of numbers. What are more diuers them a ſquare number and a cube number. And yet is 64. in diuers reſpectes a number both ſquare and cube. In reſpect of 8. to be his roote, it is a ſquare number: for 8. times 8. is 64. and in reſpect of 4. to be his roote, it is a cube number for 4. times 4. fower times is 64: ſo in diuers reſpectes it is both, without any abſurditie at all. Likewiſe this number 6. in diuers reſpectes, is a number on the on ſide longer, and alſo a trianguler number: which yet are diuers and diſtinct kindes of numbers. For 6. deſcribed by his vnities reſembling a figure of lengthe and breadthe hauing two ſides, namely 2. and 3. is a plaine or ſuperficiall number of one ſide longer. And if the ſame 6, be ſo deſcribed by his vnities, that it repreſenteth the figure of a triangle, then is it and beareth it the name of a triangler figure: as here ye may ſee the forme of either. And if ye extende in deſcription therof all his vnities in length onely, ſo is 6 alſo a lineall number. So you ſee 6 in diuers reſpectes is a lineall number, a number on the one ſide longer, and alſo a trigonall or trianguler number, and yet therby no inconuenience at all. And why may not likewiſe one and the ſelfe number in diuers reſpectes be accompted a number both euenly euen, and euenly odde? Yea *Euclide* him ſelfe doth moſt manifeſtly proue the ſame, and in the ſame wordes, if it be diligently wayed, in his ninth booke. For he ſayth, that all numbers being double in continuall courſe from the number 2. be euenly euen numbers only: and agayne all numbers whoſe halues are odde, are euenly odde numbers only: and that number which neither is duple from the number two nor hath to his halfe an odde number is a number euenly euen and a number euenly odde. What in this can be ſpokē more playnely? So that by *Euclide* it is no inconuenience that on number, as 12, for example, in diuers reſpectes ſhould be both a number euenly euen, and alſo a number euenly odde. In reſpect that 6. an euen number meaſureth 12. by 2. an euen number, 12 is a number euenly euen: and in reſpect that 4. an euen number meaſureth 12. by 3. an odde number, 12. is a number euenly odde. And thus iudge ye of all others like.

One number in diuers reſpectes may be of diuers kindes of numbers.

There is alſo an other definition geuen of this kinde of number by *Boetius* and others commonly which is thus.

Boetius definition of a number euenly euē, and euenly od.

A number euenly euen, and euenly odde is that, which may be deuided into two equall partes, and eche of them may agayne be deuided into two equall partes: and ſo forth. But this diuiſion is at lenghth ſtayd, and continueth not till it come to vnitie. As for example 48: which may be deuided into two equall partes, namely, into 24. and 24. Agayne 24. which is on of the partes may be deuided into two equall partes 12. and 12. Agayne 12, into 6. and 6. And agayne 6 may be deuided into two equall partes, into 3. and 3: but 3. cannot be deuided into two equall partes. Wherefore the deuiſion there ſtayeth: and continueth not till it come to vnitie as it did in theſe numbers which are euenly euen only.

The eleuenth definition.

11 *A number odly odde is that, which an odde number doth meaſure by an odde number.*

As 25, which 5, an odde number, meaſureth by an odde number, namely, by 5. Fiue times fiue is 25: Likewiſe 21. whom 7. an odde number doth meaſure by 3, which is likewiſe an odde number. Three times 7. is 21.

Flußates.

Fluſſatus geueth this definition following of this kinde of number, which is all one in ſubſtance with the former definition.

An other definition.

A number odly odde, is that, which onely an odde number doth meaſure.

As 15. for no number meaſureth 15. but onely 5. and 3: alſo 25: none meaſureth it but onely 5. which is an odde number, and ſo of others.

The twelfth definition.

12 *A prime (or firſt) number is that, which onely vnitie doth meaſure.*

Prime numbers called incompoſed numbers.

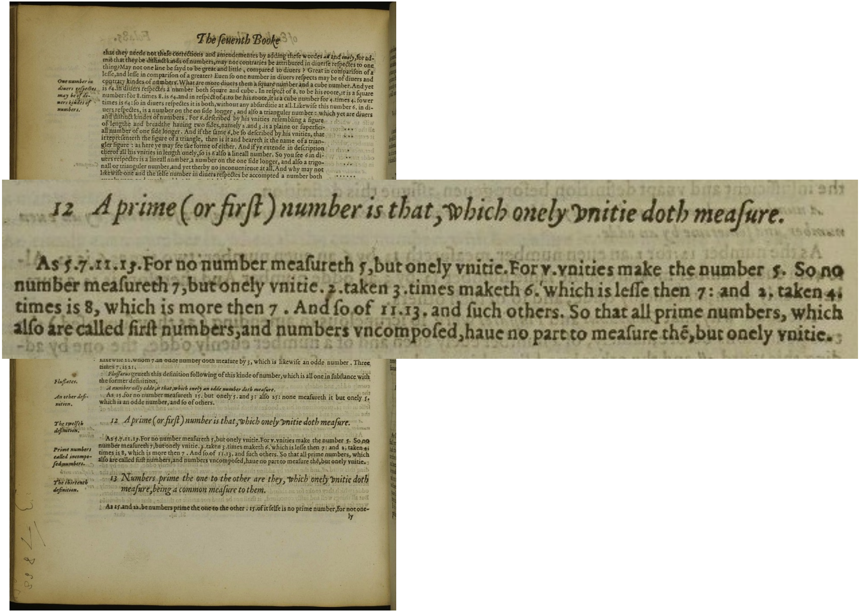
As 5.7.11.13. For no number meaſureth 5, but onely vnitie. For v. vnities make the number 5. So no number meaſureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is leſſe then 7: and 2, taken 4. times is 8, which is more then 7. And ſo of 11.13. and ſuch others. So that all prime numbers, which alſo are called firſt numbers, and numbers vncompoſed, haue no part to meaſure thē, but onely vnitie.

The thirtenth definition.

13 *Numbers prime the one to the other are they, which onely vnitie doth meaſure, being a common meaſure to them.*

As 15. and 22. be numbers prime the one to the other. 15. of it ſelfe is no prime number, for not onely

Euclid on prime numbers



The seuenth Booke

that they neede not these corrections and amendementes by adding these wordes *all* and *onely*, for admit that they be distinct kinds of numbers, may not contraries be attributed in diuerse respectes to one thing? May not one line be sayd to be great and little, compared to diuers? Great in comparison of a lesse, and lesse in comparison of a greater? Euen so one number in diuers respects may be of diuers and contrary kindes of numbers. What are more diuers them a square number and a cube number. And yet is 64. in diuers respectes a number both square and cube. In respect of 8. to be his roote, it is a square number: for 8. times 8. is 64. and in respect of 4. to be his roote, it is a cube number for 4. times 4. fower times is 64: so in diuers respectes it is both, without any absurditie at all. Likewise this number 6. in diuers respectes, is a number on the on side longer, and also a trianguler number: which yet are diuers and distinct kindes of numbers. For 6. described by his vnities resembling a figure of lengthe and breadthe hauing two sides, namely 2. and 3. is a plaine or superficiall number of one side longer. And if the same 6, be so described by his vnities, that it representeth the figure of a triangle, then is it and beareth it the name of a triangler figure: as here ye may see the forme of either. And if ye extende in description therof all his vnities in length onely, so is 6 also a lineall number. So you see 6 in diuers respectes is a lineall number, a number on the one side longer, and also a trigonall or trianguler number, and yet therby no inconuenience at all. And why may not likewise one and the selfe number in diuers respectes be accompted a number both

One number in diuers respectes may be of diuers kindes of numbers.

12 *A prime (or first) number is that, which onely vnitie doth measure.*

As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2, taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

likewise 21. whom 7. an odde number doth measure by 3, which is likewise an odde number. Three times 7. is 21.

Flussatus geueth this definition following of this kinde of number, which is all one in substance with the former definition.

Flussates.

A number oddly odde, is that, which onely an odde number doth measure.

An other definition.

As 15. for no number measureth 15. but onely 5. and 3: also 25: none measureth it but onely 5. which is an odde number, and so of others.

The twelfth definition.

12 *A prime (or first) number is that, which onely vnitie doth measure.*

As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2, taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

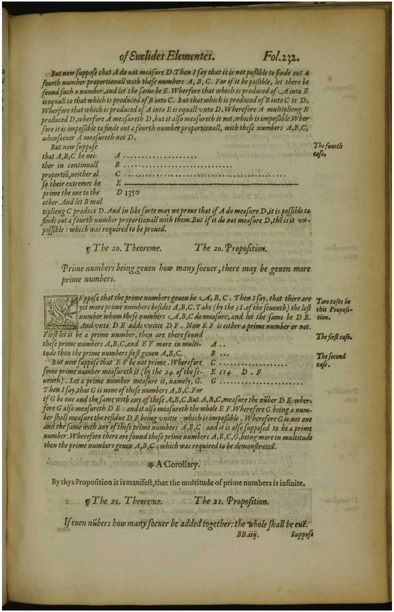
Prime numbers called incomposed numbers.

The thirtenth definition.

13 *Numbers prime the one to the other are they, which onely vnitie doth measure, being a common measure to them.*

As 15. and 22. be numbers prime the one to the other. 15. of it selfe is no prime number, for not onely

Euclid on prime numbers (Proposition IX.20)



of Euclides Elementes. Fol.232.

But now ſuppoſe that A do not meaſure D. Then I ſay that it is not poßible to finde out a fourth number proportionall with theſe numbers A, B, C. For if it be poßible, let there be found ſuch a number, and let the ſame be E. Wherfore that which is produced of A into E is equall to that which is produced of B into C. But that which is produced of B into C is D. Wherfore that which is produced of A into E is equall vnto D. Wherefore A multiplieng E produced D, wherfore A meaſureth D, but it alſo meaſureth it not, which is impoßible. Wherfore it is impoßible to finde out a fourth number proportionall, with theſe numbers A, B, C, whenſoeuer A meaſureth not D.

The fourth caſe.

But now ſuppoſe that A, B, C be neither in continuall proportiõ, neither alſo their extremes be prime the one to the other. And let B multiplieng C produce D. And in like ſorte may we proue that if A do meaſure D, it is poßible to finde out a fourth number proportionall with them. But if it do not meaſure D, thẽ is it vnpoßible : which was required to be proued.

A ....
B ....
C ....
E ———————————————
D 1350

¶ The 20. Theoreme. The 20. Propoſition.

Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

Two caſes in this Propoſition.

SVppoſe that the prime numbers geuen be A, B, C. Then I ſay, that there are yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the leſt number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not.

The firſt caſe.

Firſt let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

The ſecond caſe.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nũber D E: wherfore G alſo meaſureth D E : and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie : which is impoßible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C : and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C : which was required to be demonſtrated.

A ..
B ...
C ....
E 114 D . F
G ....

✻ A Corollary.

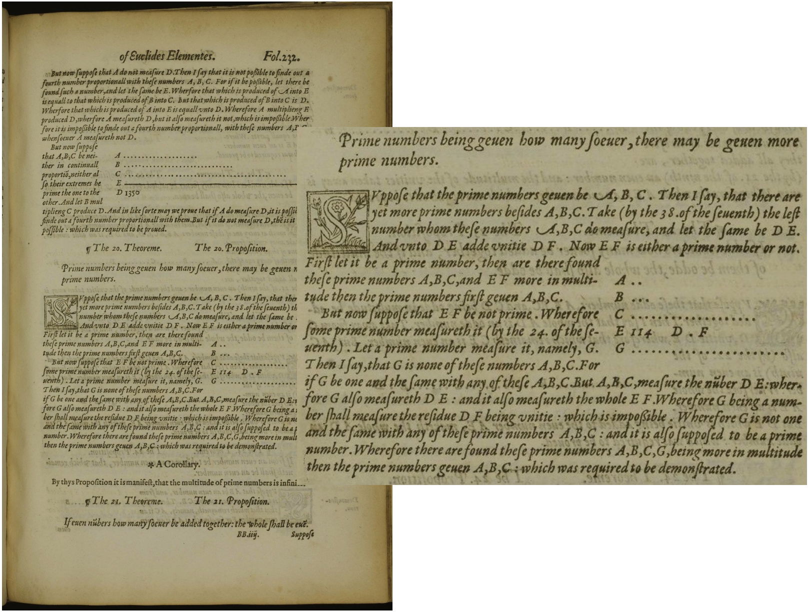
By thys Propoſition it is manifeſt, that the multitude of prime numbers is infinite.

¶ The 21. Theoreme. The 21. Propoſition.

If euen nũbers how many ſoeuer be added together: the whole ſhall be euẽ.

BB.iiij. Suppoſe

Euclid on prime numbers (Proposition IX.20)



of Euclides Elementes. Fol.232.

But now ſuppoſe that A do not meaſure D. Then I ſay that it is not poßible to finde out a fourth number proportionall with theſe numbers A, B, C. For if it be poßible, let there be found ſuch a number, and let the ſame be E. Wherfore that which is produced of A into E is equall to that which is produced of B into C. But that which is produced of B into C is D. Wherfore that which is produced of A into E is equall vnto D. Wherefore A multiplieng E produced D, wherfore A meaſureth D, but it alſo meaſureth it not, which is impoßible. Wherfore it is impoßible to finde out a fourth number proportionall, with theſe numbers A, B, C whenſoeuer A meaſureth not D.

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B ....
C ....
E ———————————————
D 1350

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Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

Suppoſe that the prime numbers geuen be A, B, C. Then I ſay, that there are yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the leſt number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. Firſt let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nūber D E: wherfore G alſo meaſureth D E: and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie: which is impoßible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C: and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonſtrated.

A ..
B ...
C ....
E 114 D . F
G ....

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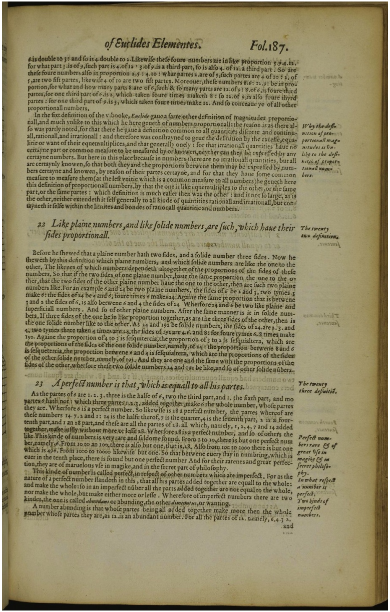
Euclid on perfect numbers



6.is double to 3: and ſo is 4.double to 2.Likewiſe theſe foure numbers are in like proportion 3.9.4.12. for what part 3.is of 9,ſuch part is 4.of 12 · 3.of 9.is a third part, ſo is alſo 4. of 12. a third part. So are theſe foure numbers alſo in proportion 2.5 : 4.10 : what partes 2.are of 5,ſuch partes are 4.of 10 : 2, of 5,are two fift partes, likewiſe 4.of 10 are two fift partes. Moreouer,theſe numbers 8.6: 12.9: be in proportion,for what and how many parts 8.are of 6,ſuch & ſo many parts are 12.of 9: 8.of 6,is foure third partes,for one third part of 6.is 2, which taken foure times maketh 8 : ſo 12.of 9,is alſo foure thyrd partes : for one third part of 9.is 3, which taken foure times make 12. And ſo conceaue ye of all other proportionall numbers.

In the ſixt definition of the v.booke, *Euclide* gaue a farre other definition of magnitudes proportionall,and much vnlike to this which he here geueth of numbers proportionall: the reaſon is as there alſo was partly noted,for that there he gaue a definition common to all quantities diſcrete and continuall,rationall,and irrationall : and therefore was conſtrayned to geue the definition by the exceſſe,equalitie or want of their equemultiplices,and that generally onely : for that irrationall quantities haue no certayne part or common meaſure to be meaſured by or knowen,neyther can they be expreſſed by any certayne numbers. But here in this place becauſe in numbers there are no irrationall quantities, but all are certaynly knowen,ſo that both they and the proportions betwene them may be expreſſed by numbers certayne and knowen, by reaſon of their partes certayne, and for that they haue ſome common meaſure to meaſure them(at the left vnitie which is a common meaſure to all numbers)he geueth here this definition of proportionall numbers,by that the one is like equemultiplex to the other,or the ſame part,or the ſame partes : which definition is much eaſier then was the other : and is not ſo large, as is the other,neither extendeth it ſelf generally to all kinde of quantities rationall and irrationall,but contayneth it ſelfe within the limites and bondes of rationall quantitie and numbers.

Why the definition of proportionall magnitudes is vnlike to the definitio of proportionall numbers.

22 *Like plaine numbers,and like ſolide numbers,are ſuch,which haue their ſides proportionall.*

The twenty two definition.

Before he ſhewed that a plaine number hath two ſides, and a ſolide number three ſides. Now he ſheweth by this definition which plaine numbers, and which ſolide numbers are like the one to the other. The likenes of which numbers dependeth altogether of the proportions of the ſides of theſe numbers. So that if the two ſides of one plaine number,haue the ſame proportion the one to the other,that the two ſides of the other plaine number haue the one to the other,then are ſuch two plaine numbers like.For an example 6 and 24 be two plaine numbers, the ſides of 6 be 2 and 3, two tymes 3 make 6: the ſides of 24 be 4 and 6, foure times 6 makes 24.Againe the ſame proportion that is betwene 3 and 2 the ſides of 6. is alſo betwene 6 and 4 the ſides of 24 Wherfore 24 and 6 be two like plaine and ſuperficiall numbers. And ſo of other plaine numbers. After the ſame manner is it in ſolide numbers. If three ſides of the one be in like proportion together,as are the three ſides of the other,then is the one ſolide number like to the other. As 24 and 192 be ſolide numbers, the ſides of 24.are 2.3. and 4, two tymes three taken 4 times are 24.the ſides of 192 are 4.6. and 8: for foure tymes 6. 8 times make 192. Againe the proportion of 4 to 3 is ſeſquitercia,the proportion of 3 to 2 is ſeſquialtera, which are the proportions of the ſides of the one ſolide number,namely,of 24 : the proportion betwene 8 and 6 is ſeſquetercia,the proportion betwene 6 and 4 is ſeſquialtera, which are the proportions of the ſides of the other ſolide number,namely,of 192. And they are one and the ſame with the proportions of the ſides of the other,wherfore theſe two ſolide numbers 24 and 192 be like,and ſo of other ſolide nũbers.

23 *A perfect number is that,which is equall to all his partes.*

The twenty three definitiõ.

As the partes of 6 are 1. 2. 3. three is the halfe of 6, two the third part,and 1. the ſixth part, and mo partes 6 hath not : which three partes 1.2.3. added together,make 6 the whole number, whoſe partes they are. Wherfore 6 is a perfect number. So likewiſe is 28 a perfect number, the partes whereof are theſe numbers 14. 7. 2 and 1: 14 is the halfe therof, 7 is the quarter, 4 is the ſeuenth part, 2 is a fourtenth part,and 1 an 28 part,and theſe are all the partes of 28. all which, namely, 1, 2, 4, 7 and 14 added together,make iuſtly without more or leſſe 28. Wherfore 28 is a perfect number, and ſo of others the like. This kinde of numbers is very rare and ſeldome found. From 1 to 10,there is but one perfect number,namely,6.From 10 to an 100,there is alſo but one,that is,28. Alſo from 100 to 1000 there is but one which is 496. From 1000 to 10000 likewiſe but one. So that betwene euery ſtay in numbring,which is euer in the tenth place,there is found but one perfect number And for their rarenes and great perfection,they are of maruelous vſe in magike,and in the ſecret part of philoſophy.

Perfect numbers rare & of great vſe in magike & in ſecret philoſophy.

This kinde of number is called perfect,in reſpect of other numbers which are imperfect. For as the nature of a perfect number ſtandeth in this, that all his partes added together are equall to the whole: and make the whole: ſo in an imperfect nũber all the parts added together are not equal to the whole, nor make the whole,but make either more or leſſe. Wherefore of imperfect numbers there are two kindes,the one is called *abundans* or abunding,the other *diminutus*,or wanting.

In what reſpect a number is perfect.

Two kinds of imperfect numbers.

A number abunding is that whoſe partes being all added together make more then the whole number whoſe partes they are,as 12.is an abundant number. For all the partes of 12. namely, 6.4.3 2.

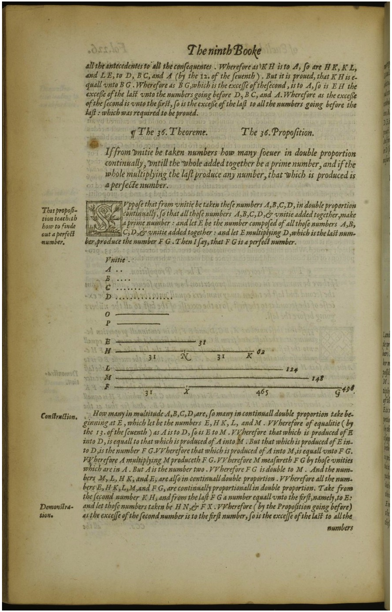
.and

Euclid on perfect numbers

23 *A perfect number is that, which is equall to all his partes.*

As the partes of 6 are 1. 2. 3. three is the halfe of 6, two the third part, and 1. the sixth part, and mo partes 6 hath not: which three partes 1.2.3. added together, make 6 the whole number, whose partes they are. Wherfore 6 is a perfect number. So likewise is 28 a perfect number, the partes whereof are these numbers 14. 7. 2 and 1: 14 is the halfe therof, 7 is the quarter, 4 is the seuenth part, 2 is a fourtenth part, and 1 an 28 part, and these are all the partes of 28. all which, namely, 1, 2, 4, 7 and 14 added together, make iustly without more or lesse 28. Wherfore 28 is a perfect number, and so of others the like. This kinde of numbers is very rare and seldome found. From 1 to 10, there is but one perfect number, namely, 6. From 10 to an 100, there is also but one, that is, 28. Also from 100 to 1000 there is but one which is 496. From 1000 to 10000 likewise but one. So that betwene euery stay in numbring, which is euer in the tenth place, there is found but one perfect number And for their rarenes and great perfection, they are of maruelous vse in magike, and in the secret part of philosophy.

Euclid on perfect numbers (Proposition IX.36)



The ninth Booke

all the antecedentes to all the consequentes. Wherefore as KH is to A, so are HK, KL, and LE, to D, BC, and A (by the 12. of the seuenth). But it is proued, that KH is equall vnto BG. Wherefore as BG, which is the excesse of the second, is to A, so is EH the excesse of the last vnto the numbers going before D, BC, and A. Wherefore as the excesse of the second is vnto the first, so is the excesse of the last to all the numbers going before the last: which was required to be proued.

¶ The 36. Theoreme. The 36. Proposition.

If from vnitie be taken numbers how many soeuer in double proportion continually, vntill the whole added together be a prime number, and if the whole multiplying the last produce any number, that which is produced is a perfecte number.

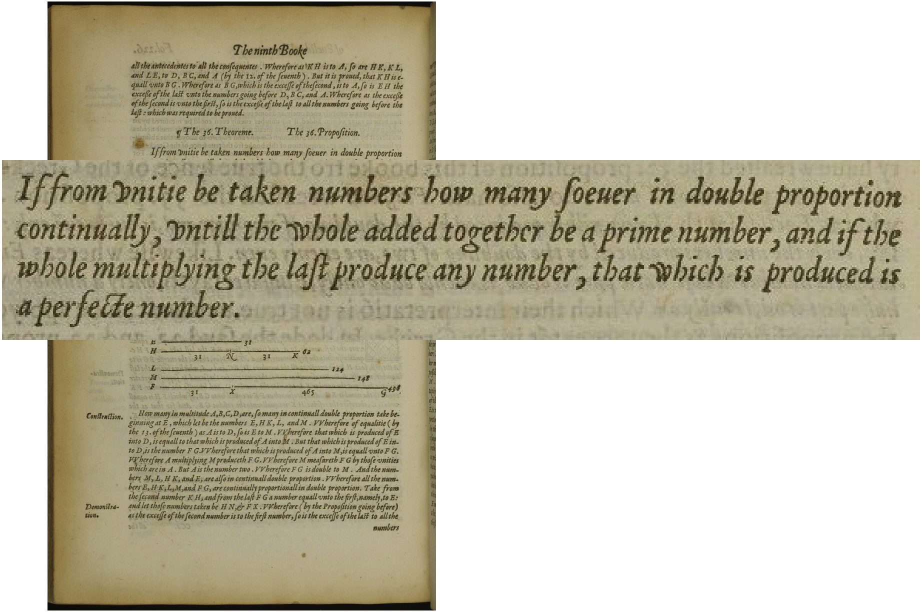
This proposition teacheth how to finde out a perfect number.

Suppose that from vnitie be taken these numbers A, B, C, D, in double proportion continually, so that all those numbers A, B, C, D, & vnitie added together, make a prime number: and let E be the number composed of all those numbers A, B, C, D, & vnitie added together: and let E multiplying D, which is the last number, produce the number FG. Then I say, that FG is a perfect number.

Construction. How many in multitude A, B, C, D, are, so many in continuall double proportion take beginning at E, which let be the numbers E, HK, L, and M. VVherefore of equalitie (by the 13. of the seuenth) as A is to D, so is E to M. VVherefore that which is produced of E into D, is equall to that which is produced of A into M. But that which is produced of E into D, is the number FG. VVherefore that which is produced of A into M, is equall vnto FG. VVherefore A multiplying M produceth FG. VVherefore M measureth FG by those vnities which are in A. But A is the number two. VVherefore FG is double to M. And the numbers M, L, HK, and E, are also in continuall double proportion. VVherefore all the numbers E, HK, L, M, and FG, are continually proportionall in double proportion. Take from the second number KH, and from the last FG a number equall vnto the first, namely, to E: and let those numbers taken be HN, & FX. VVherefore (by the Proposition going before) as the excesse of the second number is to the first number, so is the excesse of the last to all the numbers

Demonstration.

Euclid on perfect numbers (Proposition IX.36)



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Construction. How many in multitude A, B, C, D, are, so many in continuall double proportion take beginning at E, which let be the numbers E, H K, L, and M. VVherefore of equalitie (by the 13. of the seuenth) as A is to D, so is E to M. VVherefore that which is produced of E into D, is equall to that which is produced of A into M. But that which is produced of E into D, is the number F G. VVherefore that which is produced of A into M, is equall vnto F G. VVherefore A multiplying M produceth F G. VVherefore M measureth F G by those vnities which are in A. But A is the number two. VVherefore F G is double to M. And the numbers M, L, H K, and E, are also in continuall double proportion. VVherefore all the numbers E, H K, L, M, and F G, are continually proportionall in double proportion. Take from the second number K H, and from the last F G a number equall vnto the first, namely, to E: and let those numbers taken be H N, & F X. VVherefore (by the Proposition going before) as the excesse of the second number is to the first number, so is the excesse of the last to all the numbers

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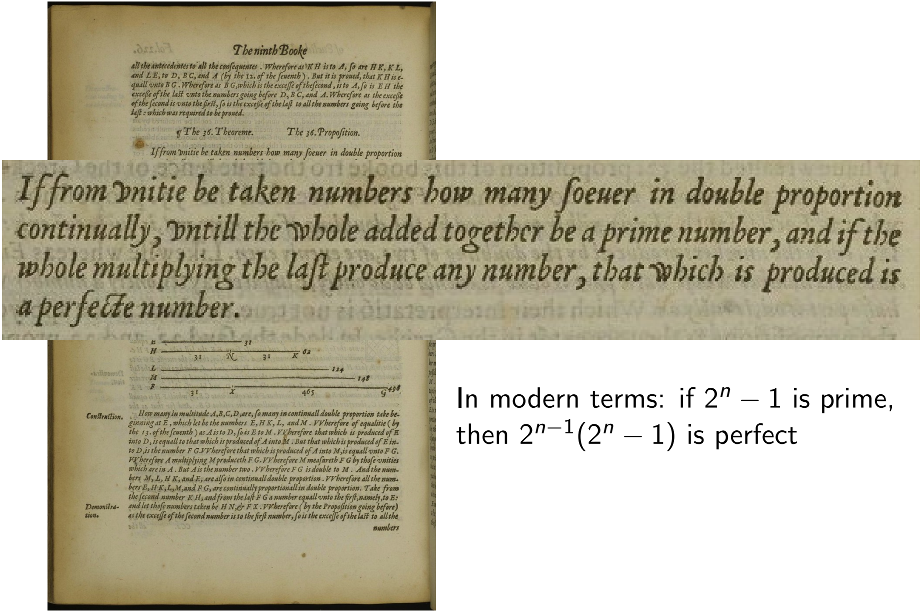

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Demonſtration. and let thoſe numbers taken be HN, & FX. VVherefore (by the Propoſition going before) as the exceſſe of the ſecond number is to the firſt number, ſo is the exceſſe of the laſt to all the numbers

In modern terms: if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect

Number theory after Euclid

Very little for many centuries...

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Restrictions on the permitted form of solutions to problems eventually gave rise to the notion of Diophantine equations

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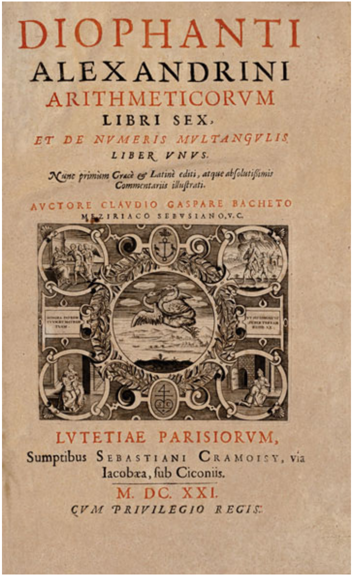
These works were unknown in Europe until the 19th century

17th-century number theory


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque abſolutiſſimis
Commentariis illuſtrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
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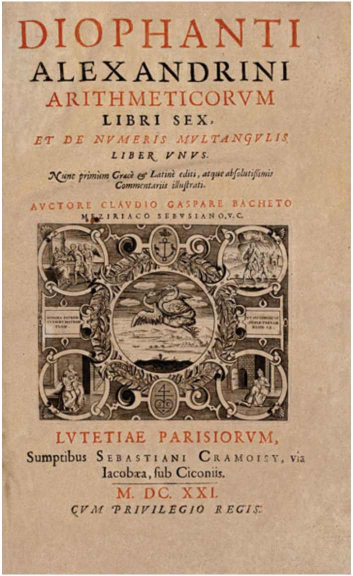
Bachet's Latin edition of Diophantus' *Arithmetica* (1621)*

*Claude Gaspard Bachet de Méziriac (1581—1638)

17th-century number theory



Bachet's Latin edition of Diophantus' *Arithmetica* (1621)*

*Claude Gaspard Bachet de Méziriac (1581—1638)

Pierre de Fermat owned a 1637 edition, which he studied and annotated

Fermat on number theory

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Published nothing — had to be exhorted to write his ideas down

(See *Mathematics emerging*, §§6.1–6.3)

The ‘Last Theorem’

Arithmetica Problem II.8 concerns the splitting of a given square number into two other squares

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Fermat’s marginal note:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

(See: Simon Singh, *Fermat’s Last Theorem*, Fourth Estate, 1998)

Perfect numbers

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Mersenne (1644): if $p \leq 257$ and $2^p - 1$ is prime then p is one of 2, 3, 5, 7, 13, 17, 67 (a misprint for 61 perhaps?), 127, 257. Not quite right: $2^{89} - 1$, $2^{107} - 1$ are prime and $2^{257} - 1$ is composite.

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NB. 51 Mersenne primes are currently known, the largest being $2^{82,589,933} - 1$ (found in June 2019)

17th-century attitudes to number theory

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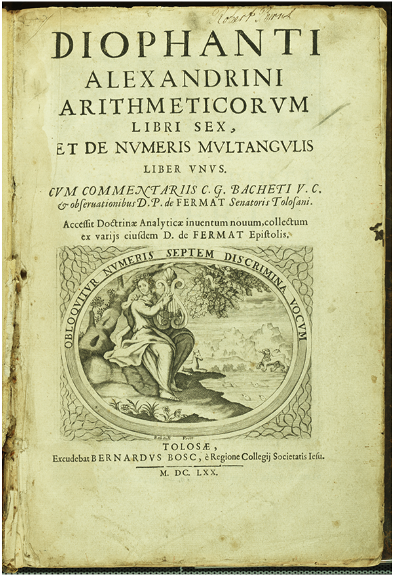
Huygens to Wallis:

> *There is no lack of better topics for us to spend our time on* ...

The ‘rebirth’ of number theory



1670 edition of Bachet, published by Samuel Fermat, including his father’s notes

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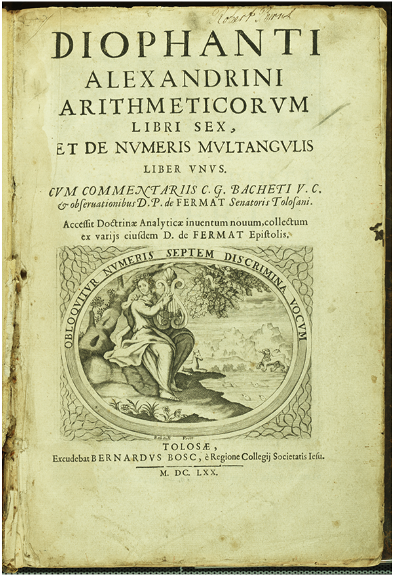

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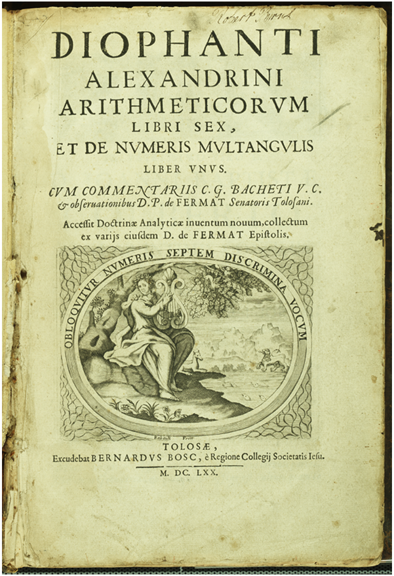
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Number theory was ‘reborn’ from the attempts of Euler (and later Lagrange and Legendre) to fill the gaps left by Fermat

Euler on number theory

Euler (1747):

> *Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. ...*
>
> *Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. ... Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import.*

19th-century number theory

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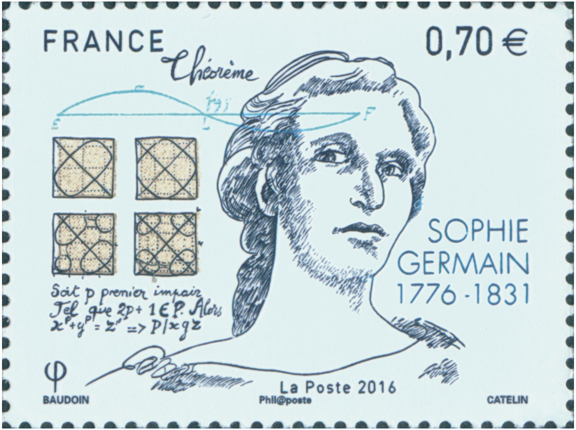
By the end of the 19th century, a new branch, analytic number theory, had also emerged (e.g., Riemann hypothesis, Prime Number Theory $\pi(x) \sim \frac{x}{\log x}$, ...)

Sophie Germain



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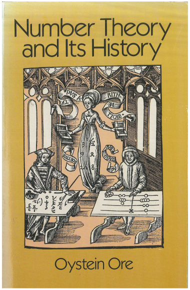
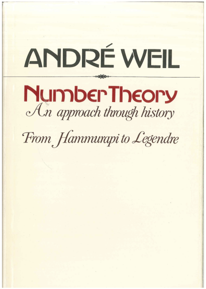
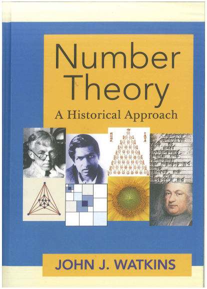
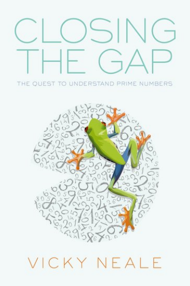
The history of number theory









Leonard Eugene Dickson, *History of the theory of numbers*, 3 vols., Carnegie Institution of Washington, 1919–1923: I, II, III