BO1 History of Mathematics
Lecture XIII
Complex analysis

MT 2021 Week 7

Summary

Part 1

- Complex numbers: validity and representation
- Substitution of complex values for real

Part 2

- Cauchy's contributions
- Riemann
- What *is* an analytic function?

Early ideas about complex numbers

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Early ideas about complex numbers

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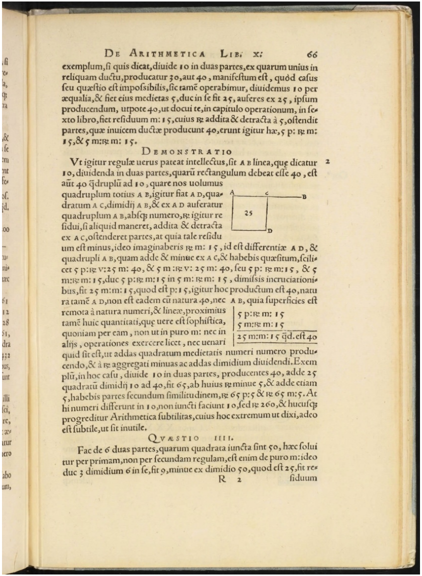
But:

> *For the most part such roots were ignored: negative roots were described merely as 'false', but complex roots as 'impossible'.* (*Mathematics emerging*, p. 459.)

Cardano and complex numbers



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DEMONSTRATIO

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat esse 40, est aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, seu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamẽ A D, non est eadem cũ natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamẽ huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exemplũ, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam 5, habebis partes secundum similitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed ℞ 260, & hucusq; progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̃d. est 40

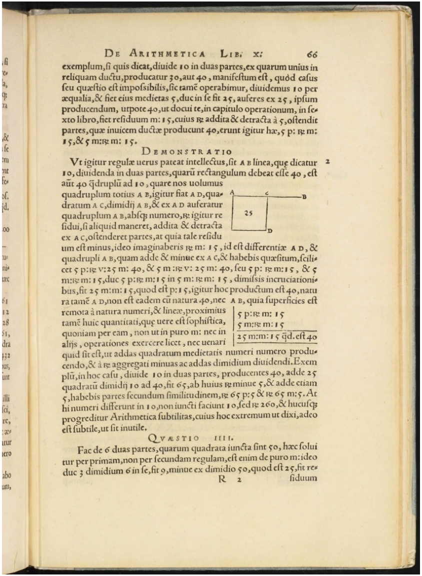
QVÆSTIO IIII.

Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc soluitur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

R 2

Problem: find two numbers that add to 10 and multiply to 40,

Cardano and complex numbers



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5 p: ℞ m: 15
5 m: ℞ m: 15
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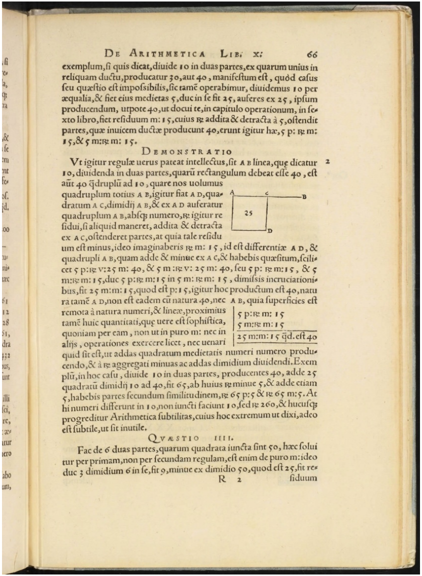
Quæstio IIII.

Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc soluitur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

R 2

Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type ‘square plus number equals thing’

Cardano and complex numbers



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5 p: ℞ m: 15
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Fac de 6 duas partes, quarum quadrata iuncta ſint 50, hæc ſoluitur per primam, non per ſecundam regulam, eſt enim de puro m: ideo duc 3 dimidium 6 in ſe, fit 9, minue ex dimidio 50, quod eſt 25, fit reſiduum

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Cardano noted that $5+\sqrt{-15}$ and $5-\sqrt{-15}$ solve the problem, "dismissis incruciationibus",

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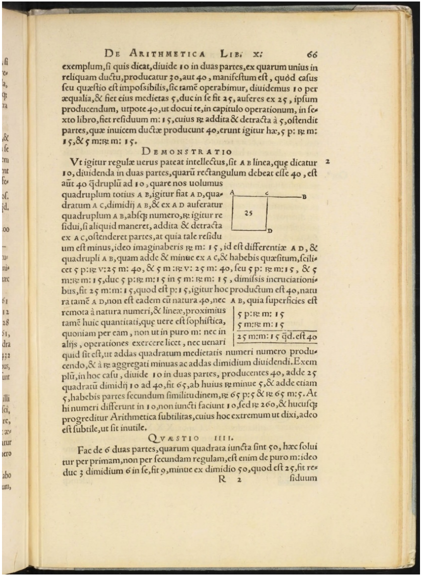

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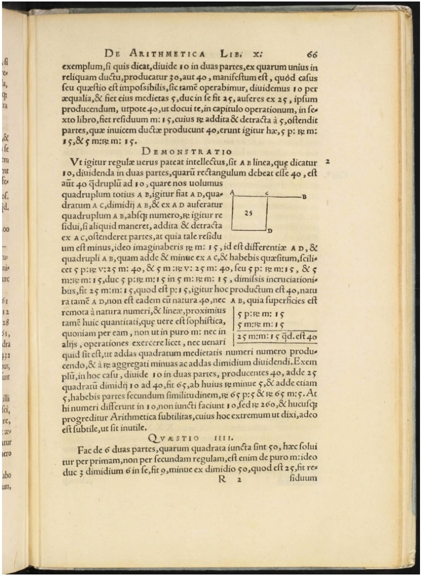

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Demonstratio

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarū rectangulum debeat esse 40, est aūt 40 q̄druplū ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absque numero, R igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis R m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: R v: 25 m: 40, & 5 m: R v: 25 m: 40, seu 5 p: R m: 15, & 5 m: R m: 15, duc 5 p: R m: 15 in 5 m: R m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamē A D, non est eadem cū natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamē huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à R aggregati minuas ac addas dimidium diuidendi. Exemplū, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratū dimidij 10 ad 40, fit 65, ab huius R minue 5, & adde etiam 5, habebis partes secundum similitudinem, R 65 p: 5 & R 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed R 260, & hucusque progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: R m: 15
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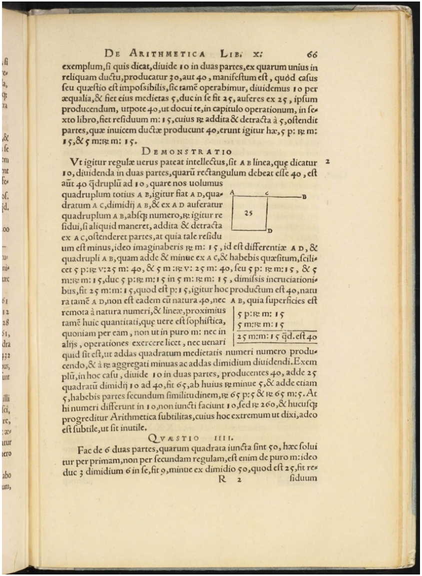
Cardano noted that $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ solve the problem, "dismissis incruciationibus", meaning

"putting aside mental tortures",

or

"the cross-multiples having canceled out",

Cardano and complex numbers



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Fac de 6 duas partes,quarum quadrata iuncta ſint 50, hæc ſoluitur per primam,non per ſecundam regulam,eſt enim de puro m: ideo duc 3 dimidium 6 in ſe,fit 9,minue ex dimidio 50,quod eſt 25,fit reſiduum

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or
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or
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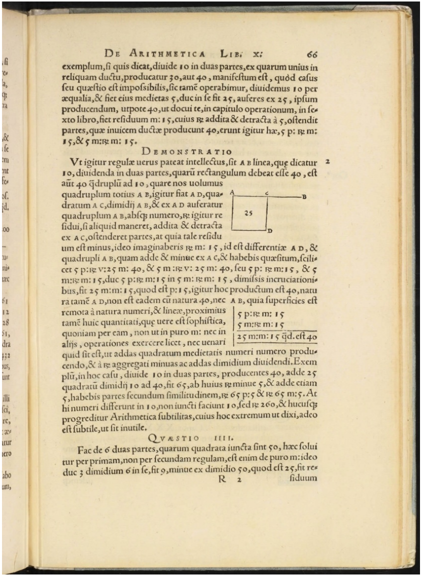
Cardano and complex numbers



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Demonstratio

Vt igitur regulæ uerus pateat intellectus, ſit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat eſſe 40, eſt aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, abſq; numero, ℞ igitur reſidui, ſi aliquid maneret, addita & detracta ex A C, oſtenderet partes, at quia tale reſiduum eſt minus, ideo imaginaberis ℞ m: 15, id eſt differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæſitum, ſcilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, ſeu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimiſsis incruciationibus, fit 25 m: m: 15, quod eſt p: 15, igitur hoc productum eſt 40, natura tamẽ A D, non eſt eadem cũ natura 40, nec A B, quia ſuperficies eſt remota à natura numeri, & lineæ, proximius tamẽ huic quantitati, quę uere eſt ſophiſtica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid ſit eſt, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exemplũ, in hoc caſu, diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam 5, habebis partes ſecundum ſimilitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, ſed ℞ 260, & hucuſq; progreditur Arithmetica ſubtilitas, cuius hoc extremum ut dixi, adeo eſt ſubtile, ut ſit inutile.

5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̃d. eſt 40

Qvæstio IIII.

Fac de 6 duas partes, quarum quadrata iuncta ſint 50, hæc ſoluitur per primam, non per ſecundam regulam, eſt enim de puro m: ideo duc 3 dimidium 6 in ſe, fit 9, minue ex dimidio 50, quod eſt 25, fit reſiduum

R 2

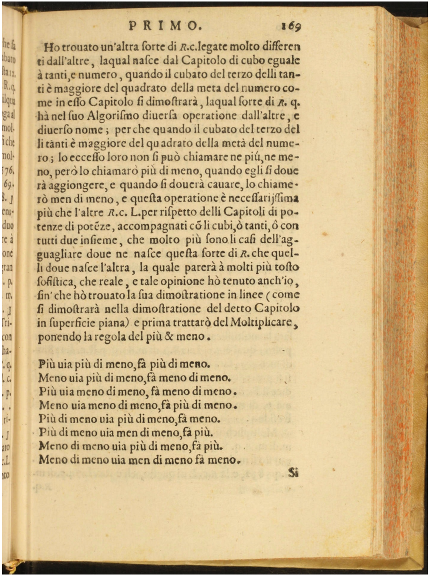
Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type 'square plus number equals thing'

Cardano noted that $5+\sqrt{-15}$ and $5-\sqrt{-15}$ solve the problem, "dismissis incruciationibus", meaning

"putting aside mental tortures",
or
"the cross-multiples having canceled out",
or
"the imaginary part being lost"

But regarded such ideas as absurd and useless

Bombelli and complex numbers


PRIMO. 169

Ho trouato un'altra ſorte di R.c.legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; per che quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo ecceſſo loro non ſi può chiamare ne più, ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſariſſima più che l'altre R.c. L.per riſpetto delli Capitoli di potenze di potẽze, accompagnati cõ li cubi, ò tanti, ò con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, fin'che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

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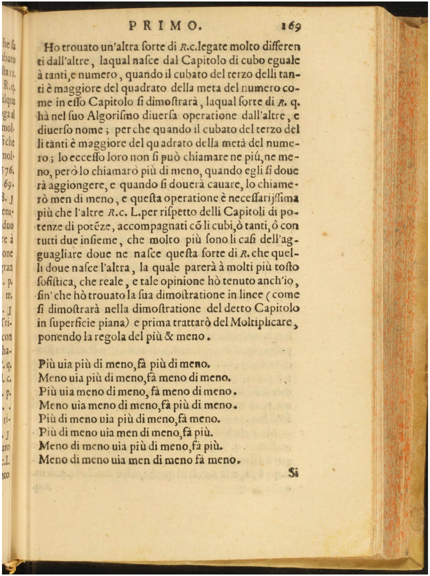
Si

"Another sort of cube root much different from the former . . ."

Bombelli and complex numbers



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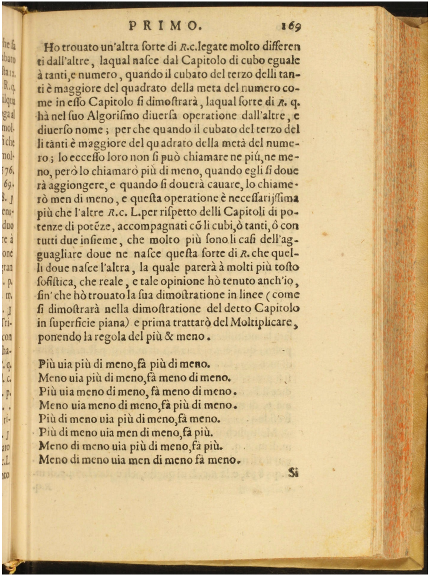
Bombelli and complex numbers



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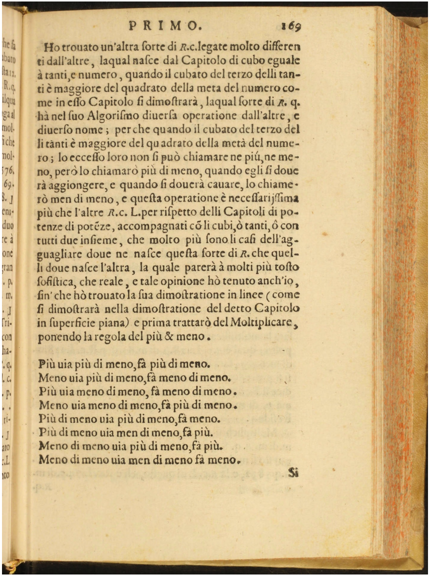
But complex numbers were not admitted as solutions of equations — they could appear in calculations, provided they cancelled out by the end

Bombelli and complex numbers



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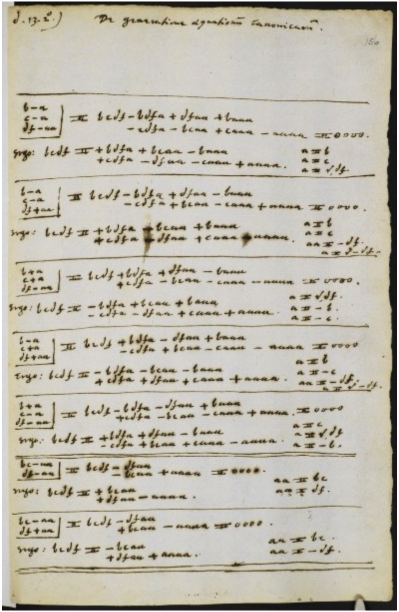
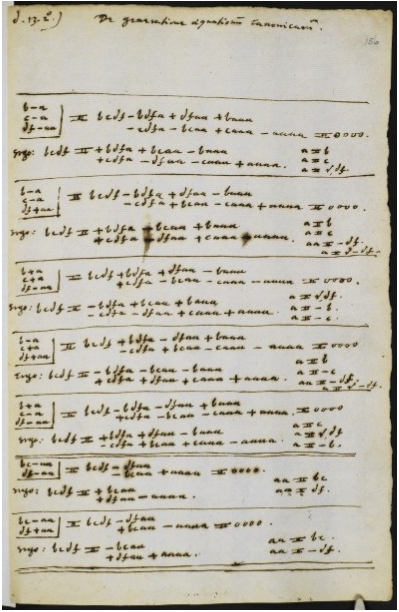
Complex numbers justified through practical use?

Harriot and complex numbers



Add MS 6783 f. 156

Harriot and complex numbers



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Unpublished manuscripts contain systematic treatment of complex roots of equations

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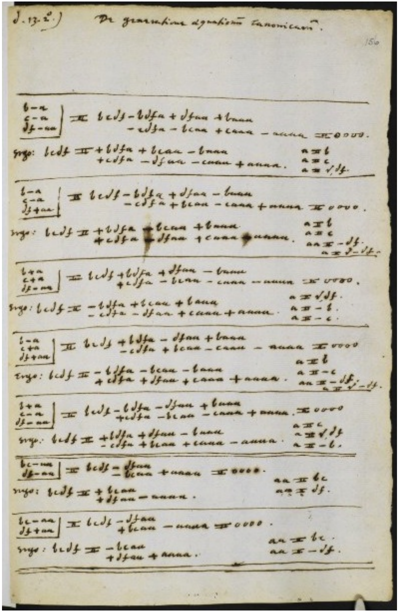

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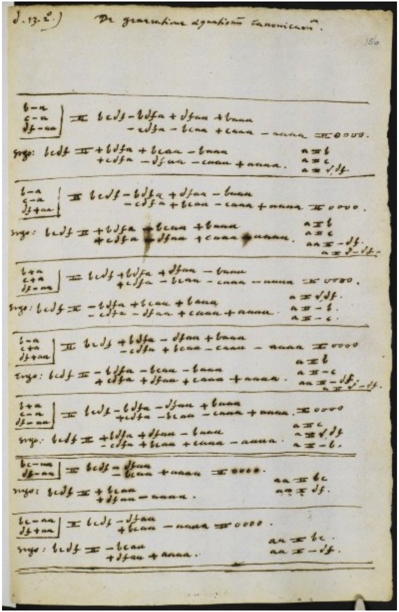

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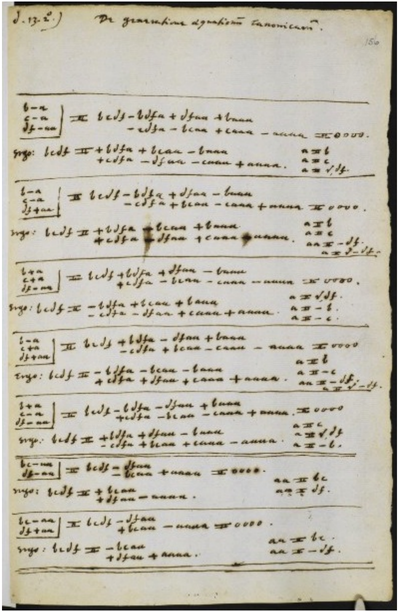

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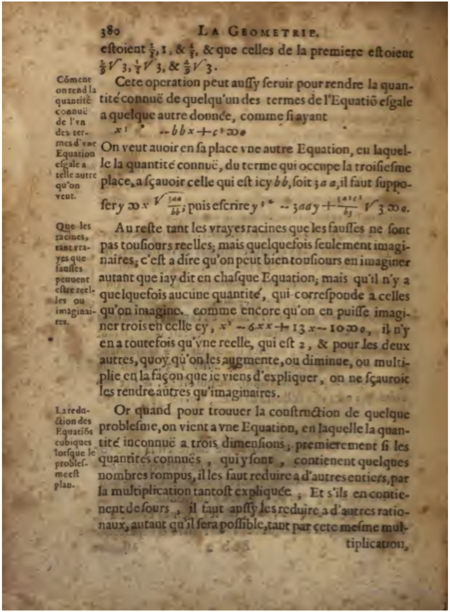
Cf. Harriot's *Artis analyticae praxis* (1931), pp. 14–15; see:

Muriel Seltman & Robert Goulding, *Thomas Harriot's Artis analyticae praxis: an English translation with commentary*, Springer, 2007

Descartes and 'imaginaries'



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Cõment on rend la quantité connuë de l'vn des termes d'vne Equation esgale a telle autre qu'on veut.

Cete operation peut aussy seruir pour rendre la quantité connuë de quelqu'un des termes de l'Equatiõ esgale a quelque autre donnée, comme si ayant

$$x^3 * - bbx + c^3 \infty 0$$

On veut auoir en sa place vne autre Equation, en laquelle la quantité connuë, du terme qui occupe la troisiesme place, a sçauoir celle qui est icy bb, soit $3aa$, il faut supposer $y \infty x\sqrt{\frac{3aa}{bb}}$; puis escrire $y^3 * - 3aay + \frac{3a^3c^3}{b^3}\sqrt{3} \infty 0$.

Que les racines, tant vrayes que fausses peuuent estre reelles ou imaginaires.

Au reste tant les vrayes racines que les fausses ne sont pas tousiours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousiours en imaginer autant que iay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine. comme encore qu'on en puisse imaginer trois en celle cy, $x^3 - 6xx + 13x - 10 \infty 0$, il n'y en a toutefois qu'vne reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que ie viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires.

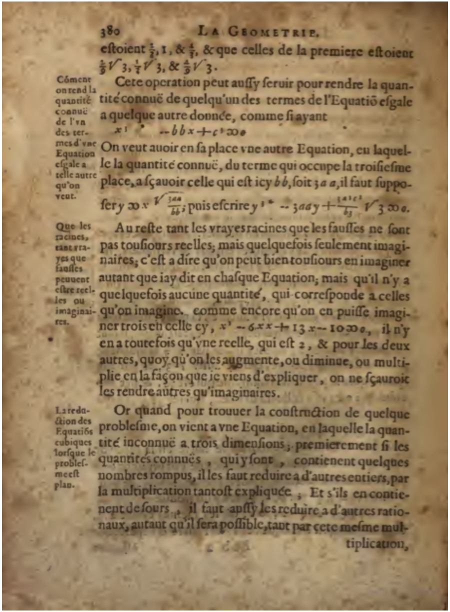
La reduction des Equatiõs cubiques lorsque le problesme est plan.

Or quand pour trouuer la construction de quelque problesme, on vient a vne Equation, en laquelle la quantité inconnuë a trois dimensions; premierement si les quantités connuës, qui y sont, contienent quelques nombres rompus, il les faut reduire a d'autres entiers, par la multiplication tantost expliquée; Et s'ils en contienent de sours, il faut aussy les reduire a d'autres rationaux, autant qu'il sera possible, tant par cete mesme multiplication,

La géométrie (1637):

introduced the term 'imaginaire'

Descartes and 'imaginaries'



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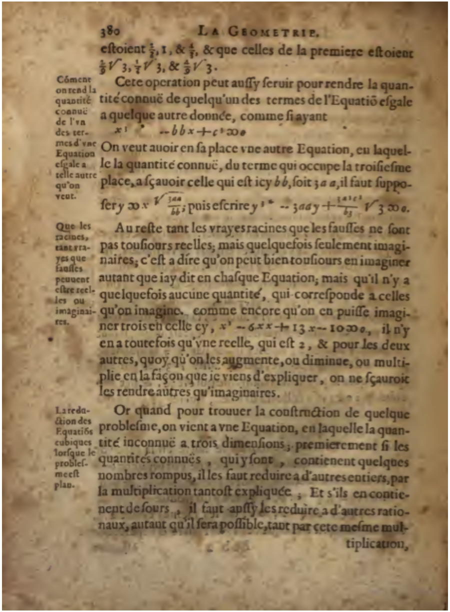

380 LA GEOMETRIE.

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$$x^3 \; * \; - bbx + c^3 \infty 0$$

On veut auoir en ſa place vne autre Equation, en laquelle la quantité connuë, du terme qui occupe la troiſieſme place, a ſçauoir celle qui eſt icy bb, ſoit $3aa$, il faut ſuppoſer $y \infty x\sqrt{\frac{3aa}{bb}}$; puis eſcrire $y^3 \; * \; - 3aay + \frac{3a^3c^3}{b^3}\sqrt{3} \infty 0$.

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La reduction des Equatiõs cubiques lorſque le probleſme eſt plan.

Or quand pour trouuer la conſtruction de quelque probleſme, on vient a vne Equation, en laquelle la quantité inconnuë a trois dimenſions; premierement ſi les quantités connuës, qui y ſont, contienent quelques nombres rompus, il les faut reduire a d'autres entiers, par la multiplication tantoſt expliquée; Et s'ils en contienent de ſours, il faut auſſy les reduire a d'autres rationaux, autant qu'il ſera poſſible, tant par cete meſme multiplication,

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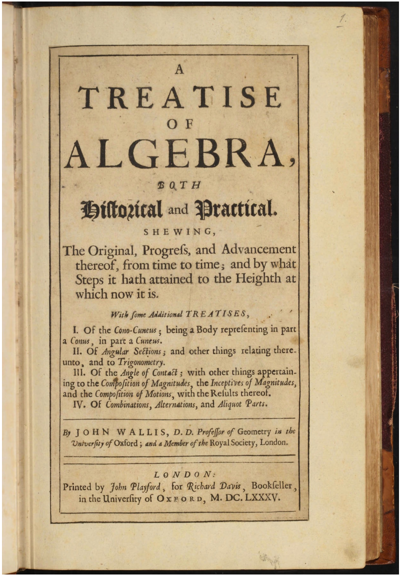
Didn’t regard them as numbers

Ideas about complex numbers in the later 17th century

John Wallis, *A treatise of algebra* (1685): complex numbers based on insights derived from

- Euclidean geometry
- trigonometry
- properties of conics

(See: *Mathematics emerging*, §15.1.1.)



A
TREATISE
OF
ALGEBRA,
BOTH
Historical and Practical.
SHEWING,
The Original, Progreſs, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With ſome Additional TREATISES,

I. Of the Cono-Cuneus; being a Body repreſenting in part a Conus, in part a Cuneus.
II. Of Angular Sections; and other things relating thereunto, and to Trigonometry.
III. Of the Angle of Contact; with other things appertaining to the Compoſition of Magnitudes, the Inceptives of Magnitudes, and the Compoſition of Motions, with the Reſults thereof.
IV. Of Combinations, Alternations, and Aliquot Parts.

By JOHN WALLIS, D.D. Profeſſor of Geometry in the Univerſity of Oxford; and a Member of the Royal Society, London.

LONDON:
Printed by John Playford, for Richard Davis, Bookſeller, in the Univerſity of OXFORD, M. DC. LXXXV.

Wallis: justification of imaginary numbers

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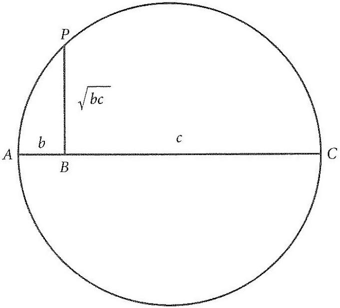
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16. The area in question (assumed to be a square) might therefore be viewed as having side $\sqrt{-16}$.

(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 184–185)

Wallis: imaginary numbers as geometric means



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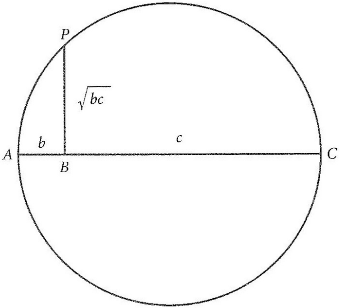
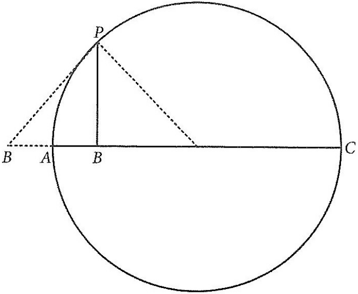


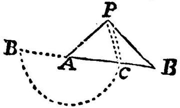


(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 185–186)

“A new Impossibility in Algebra”

John Wallis, *A treatise of algebra*, p. 267 ‘Of negative squares’:
... requires a new Impossibility in Algebra

Suppoſe again, AP = 15, PC = 12, (and therefore AC = √: 225 − 144: = √81 = 9,) PB = 20 (and therefore BC = √: 400 − 144: = √256 = + 16, or − 16:) Then is AB = 9 + 16 = 25, or AB = 9 − 16 = − 7. The one Affirmative, the other Negative. (The ſame values would be, but with contrary Signs, if we take AC = √81 = −9: That is, AB = −9 + 16 = +7, AB = −9 − 16 = −25.)



⋮

Which gives indeed (as before) a double value of AB, √175, + √−81, and √175, − √−81: But ſuch as requires a new Impoſſibility in Algebra, (which in Lateral Equations doth not happen;) not that of a Negative Root, or a Quantity leſs than nothing; (as before,) but the Root of a Negative Square. Which in ſtrictneſs of ſpeech, cannot be: ſince that no Real Root (Affirmative or Negative,) being Multiplied into itſelf, will make a Negative Square.

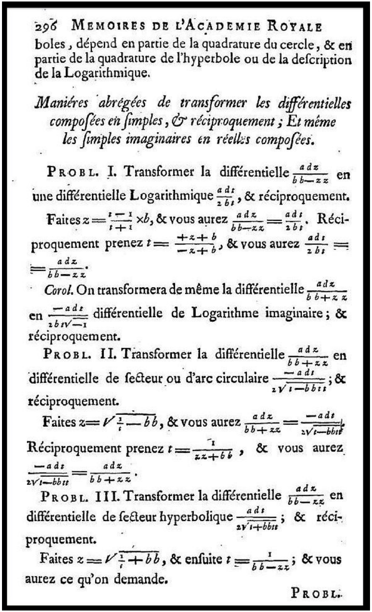
Complex numbers in the 18th century (1)

Nature remained unclear:

“that amphibian between being and not-being, which we call the imaginary root of negative unity” (Leibniz, 1702)

But complex numbers were increasingly being used . . .

Complex numbers in the 18th century (2)



boles, dépend en partie de la quadrature du cercle, & en partie de la quadrature de l'hyperbole ou de la deſcription de la Logarithmique.

Maniéres abrégées de transformer les différentielles compoſées en ſimples, & réciproquement; Et même les ſimples imaginaires en réelles compoſées.

PROBL. I. Transformer la différentielle $\frac{adz}{bb-zz}$ en une différentielle Logarithmique $\frac{adt}{2bt}$, & réciproquement.

Faites $z=\frac{t-1}{t+1}\times b$, & vous aurez $\frac{adz}{bb-zz}=\frac{adt}{2bt}$. Réciproquement prenez $t=\frac{+z+b}{-z+b}$, & vous aurez $\frac{adt}{2bt}=\frac{adz}{bb-zz}$.

Corol. On transformera de même la différentielle $\frac{adz}{bb+zz}$ en $\frac{-adt}{2bt\sqrt{-1}}$ différentielle de Logarithme imaginaire; & réciproquement.

PROBL. II. Transformer la différentielle $\frac{adz}{bb+zz}$ en différentielle de ſecteur ou d'arc circulaire $\frac{-adt}{2\sqrt{t-bbtt}}$; & réciproquement.

Faites $z=\sqrt{\frac{1}{t}-bb}$, & vous aurez $\frac{adz}{bb+zz}=\frac{-adt}{2\sqrt{t-bbtt}}$. Réciproquement prenez $t=\frac{1}{zz+bb}$, & vous aurez $\frac{-adt}{2\sqrt{t-bbtt}}=\frac{adz}{bb+zz}$.

PROBL. III. Transformer la différentielle $\frac{adz}{bb-zz}$ en différentielle de ſecteur hyperbolique $\frac{adt}{2\sqrt{t+bbtt}}$; & réciproquement.

Faites $z=\sqrt{\frac{1}{t}+bb}$, & enſuite $t=\frac{1}{bb-zz}$; & vous aurez ce qu'on demande.

PROBL.

Johann Bernoulli, 'Solution d'un problème concernant le calcul intégrale, ...', *Mémoires de l'Académie royale des sciences*, 1702:

Complex numbers in the 18th century (2)

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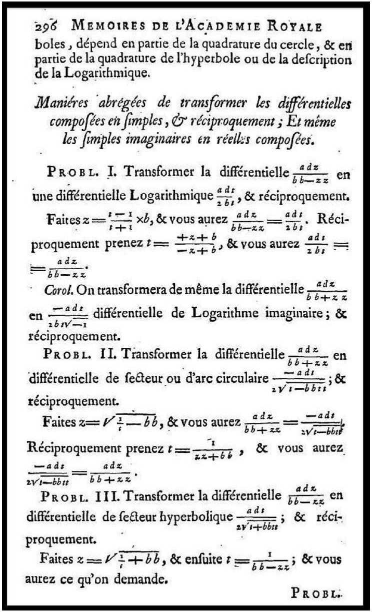
Complex numbers in the 18th century (2)



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No worries about the validity of switching between real and complex integrals

(See *Mathematics emerging*, §15.2.1)

Complex numbers in the 18th century (3)

How ÆQUATIONS *are to be ſolv'd.*

AFTER therefore in the Solution of a Queſtion you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are ſuppos'd given, are really given in Numbers, thoſe Numbers are to be ſubſtituted in their room in the Æquation, and you'll have a Numeral Æquation, whoſe Root being extracted will ſatisfy the Queſtion. As if in the Diviſion of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Caſe the Radius r is given in Numbers, and the Line q ſubtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I ſubſtitute thoſe Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line ſubtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Æquation.

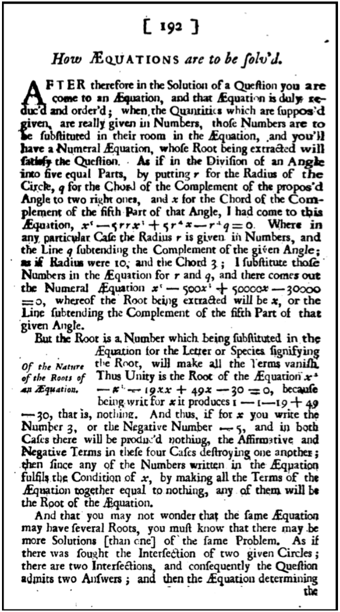
But the Root is a Number which being ſubſtituted in the Æquation for the Letter or Species ſignifying the Root, will make all the Terms vaniſh. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, becauſe being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for x you write the Number 3, or the Negative Number -5, and in both Caſes there will be produc'd nothing, the Affirmative and Negative Terms in theſe four Caſes deſtroying one another; then ſince any of the Numbers written in the Æquation fulfils the Condition of x, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the ſame Æquation may have ſeveral Roots, you muſt know that there may be more Solutions [than one] of the ſame Problem. As if there was ſought the Interſection of two given Circles; there are two Interſections, and conſequently the Queſtion admits two Anſwers; and then the Æquation determining the

Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible"

Complex numbers in the 18th century (3)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How ÆQUATIONS *are to be solv'd.*

AFTER therefore in the Solution of a Question you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are suppos'd given, are really given in Numbers, those Numbers are to be substituted in their room in the Æquation, and you'll have a Numeral Æquation, whose Root being extracted will satisfy the Question. As if in the Division of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Case the Radius r is given in Numbers, and the Line q subtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I substitute those Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line subtending the Complement of the fifth Part of that given Angle.

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And that you may not wonder that the same Æquation may have several Roots, you must know that there may be more Solutions [than one] of the same Problem. As if there was sought the Intersection of two given Circles; there are two Intersections, and consequently the Question admits two Answers; and then the Æquation determining the

Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible" — complex numbers as an indicator of real-world solvability of problems

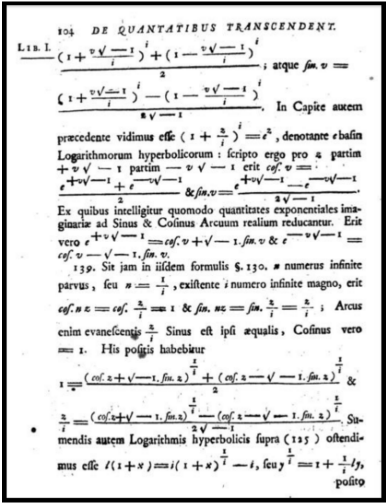
Complex numbers in the 18th century (4)

Leonhard Euler also used them freely: e.g., in *Introductio in analysin infinitorum*, 1748, §138:

$$e^{+v\sqrt{-1}} = \cos.v + \sqrt{-1}.\sin.v$$

$$e^{-v\sqrt{-1}} = \cos.v - \sqrt{-1}.\sin.v$$

(See *Mathematics emerging*, §9.2.3)



104 DE QUANTATIBUS TRANSCENDENT.

LIB. I.

$\dfrac{(1+\frac{v\sqrt{-1}}{i})^i+(1-\frac{v\sqrt{-1}}{i})^i}{2}$; atque $fin.\,v =$

$\dfrac{(1+\frac{v\sqrt{-1}}{i})^i-(1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In Capite autem

præcedente vidimus esse $(1+\frac{z}{i})^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $cof.\,v =$

$\dfrac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2}$ & $fin.\,v = \dfrac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}$.

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cofinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = cof.\,v+\sqrt{-1}.\,fin.\,v$ & $e^{-v\sqrt{-1}} = cof.\,v-\sqrt{-1}.\,fin.\,v$.

139. Sit jam in iisdem formulis §. 130. n numerus infinite parvus, seu $n = \frac{1}{i}$, existente i numero infinite magno, erit $cof.\,nz = cof.\,\frac{z}{i} = 1$ & $fin.\,nz = fin.\,\frac{z}{i} = \frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cofinus vero $= 1$. His positis habebitur

$1 = \dfrac{(cof.\,z+\sqrt{-1}.\,fin.\,z)^{\frac{1}{i}}+(cof.\,z-\sqrt{-1}.\,fin.\,z)^{\frac{1}{i}}}{2}$ &

$\dfrac{z}{i} = \dfrac{(cof.\,z+\sqrt{-1}.\,fin.\,z)^{\frac{1}{i}}-(cof.\,z-\sqrt{-1}.\,fin.\,z)^{\frac{1}{i}}}{2\sqrt{-1}}$. Sumendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, seu $y^{\frac{1}{i}} = 1+\frac{1}{i}ly$,

posito

The Fundamental Theorem of Algebra

Every polynomial equation of degree n has exactly n roots.

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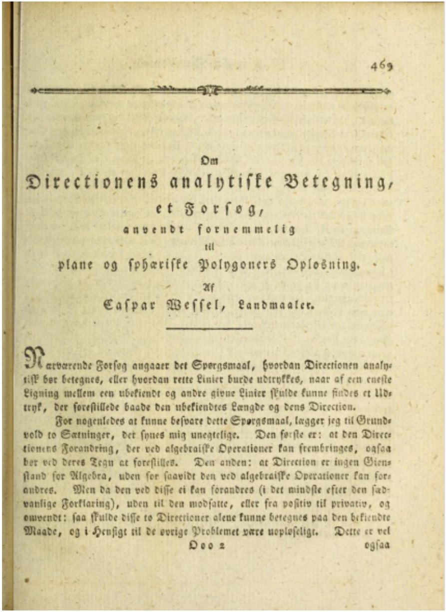
- 1821: Argand's proof appears in Cauchy's *Cours d'analyse*

New ways of viewing complex numbers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Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Opløsning.

Af

Caspar Wessel, Landmaaler.

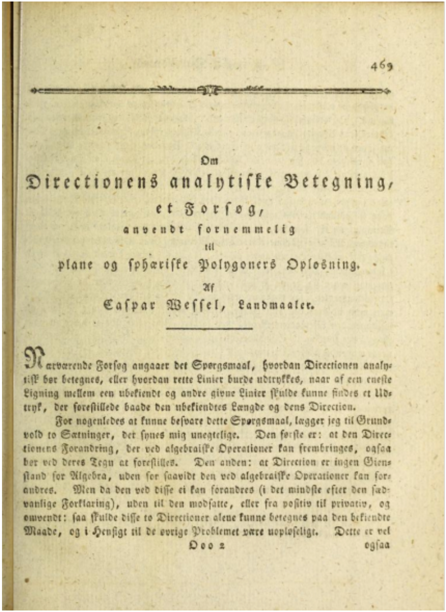
Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

For nogenledes at kunne besvare dette Spørgsmaal, lægger jeg til Grundvold to Sætninger, der synes mig unegtelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan frembringes, ogsaa bør ved deres Tegn at forestilles. Den anden: at Direction er ingen Gienstand for Algebra, uden for saavidt den ved algebraiske Operationer kan forandres. Men da den ved disse ei kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt: saa skulde disse to Directioner alene kunne betegnes paa den bekiendte Maade, og i Hensigt til de øvrige Problemet være uopløselige. Dette er vel

Ooo 2 ogsaa

Caspar Wessel, 'Om Directionens analytiske Betegning . . .' ['On the analytic representation of direction . . .'], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

New ways of viewing complex numbers



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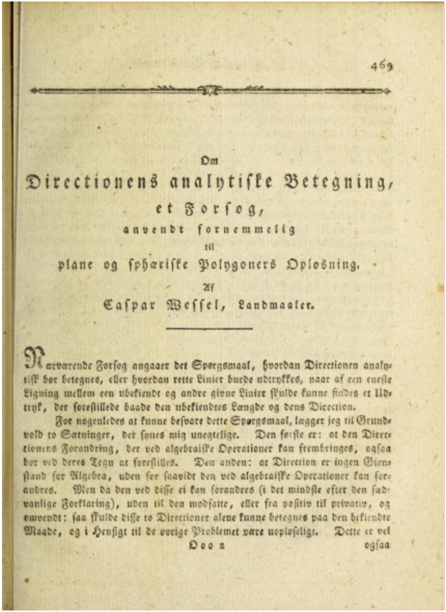
Ooo 2 ogsaa

Caspar Wessel, 'Om Directionens analytiske Betegning …' ['On the analytic representation of direction …'], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

Published in Danish

New ways of viewing complex numbers



469

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Directionens analytiske Betegning,
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plane og sphæriske Polygoners Opløsning.
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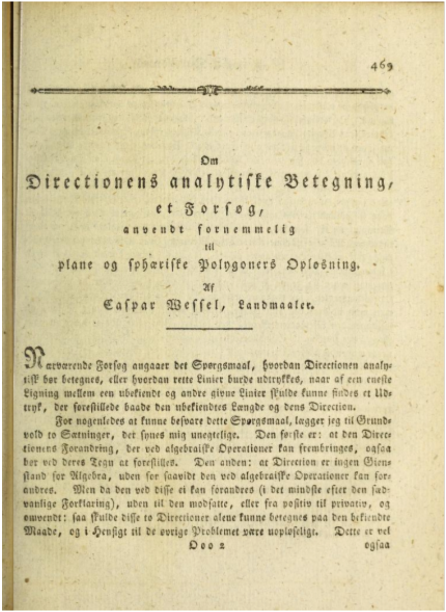

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French translation published in 1897

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ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

LES QUANTITÉS IMAGINAIRES

DANS

LES CONSTRUCTIONS GÉOMÉTRIQUES,

PAR R. ARGAND.

2e ÉDITION

PRÉCÉDÉE D'UNE PRÉFACE

PAR M. J. HOÜEL

ET SUIVIE D'UN APPENDICE

Contenant des Extraits des *Annales de Gergonne*, relatifs à la question des imaginaires.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

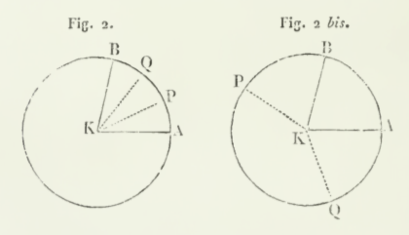
SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1874

(Tous droits réservés.)

Robert Argand, *Essay on a method of representing imaginary quantities* . . ., 1806



New ways of viewing complex numbers

Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

By WILLIAM ROWAN HAMILTON,

M. R. I. A., F. R. A. S., Hon. M. R. S. Ed. and Dub., Fellow of the American Academy of Arts and Sciences, and of the Royal Northern Antiquarian Society at Copenhagen, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

THE Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere*,) is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax; when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them; or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula: when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

Transactions of the Royal Irish Academy, 1837

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Led to the search for triples, and thence to quaternions

Part 2: Functions of a Complex Variable

Complex analysis

The origins of complex analysis may be seen in early achievements by Johann Bernoulli, Euler, and others, using complex transformations to evaluate real integrals.

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- Euler (posthumous, 1794): yes
- Laplace (1785, 1812): yes
- Poisson (1812): doubtful
- Cauchy (1814): inspired by Laplace, set to work on the problem

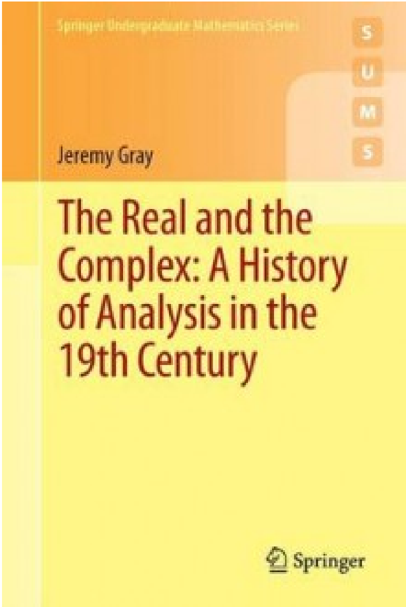
Sources for the origins of complex analysis

Secondary:

- Katz: §17.3 (3rd ed.); §22.3 (brief ed.)
- Frank Smithies: *Cauchy and the creation of complex function theory*, Cambridge University Press, 1997

Primary: as quoted by Smithies; some extracts reproduced in *Mathematics emerging*, §15.2.

Real and complex analysis united



Cauchy as 'creator' of complex analysis

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Cauchy's first 'Mémoire' (1814/1827)

MÉMOIRE

SUR

LES INTÉGRALES DÉFINIES[1].

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La solution d'un grand nombre de problèmes se réduit, en dernière analyse, à l'évaluation des intégrales définies; aussi les géomètres se sont-ils beaucoup occupés de leur détermination. On trouve, à cet égard, une foule de théorèmes curieux et utiles dans les Mémoires et le *Calcul intégral* d'Euler, dans plusieurs Mémoires de M. Laplace, dans ses *Recherches sur les approximations de certaines formules*, et dans les *Exercices de Calcul intégral* de M. Legendre. Mais, parmi les diverses intégrales obtenues par les deux premiers géomètres que je viens de citer, plusieurs ont été découvertes pour la première fois à l'aide d'une espèce d'induction fondée sur le passage du réel à l'imaginaire. Les passages de cette nature conduisent souvent d'une manière très prompte à des résultats dignes de remarque. Toutefois cette portion de la théorie est, ainsi que l'a observé M. Laplace, sujette à plusieurs difficultés. Aussi, après avoir montré, dans le calcul des fonctions génératrices, les ressources que l'Analyse peut retirer de semblables considérations, l'auteur ajoute : « On peut donc considérer ces passages comme des moyens de découvertes semblables à l'induction dont les

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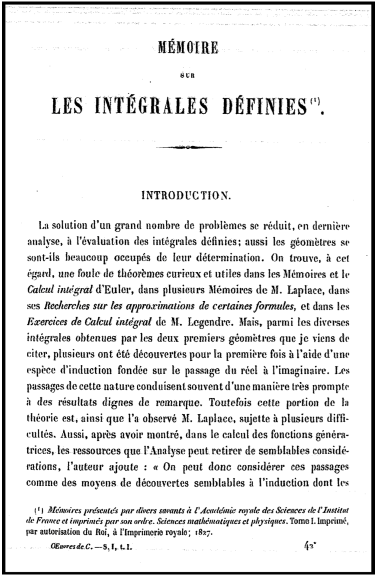

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Œuvres de C. — S. I, t. I. 42*

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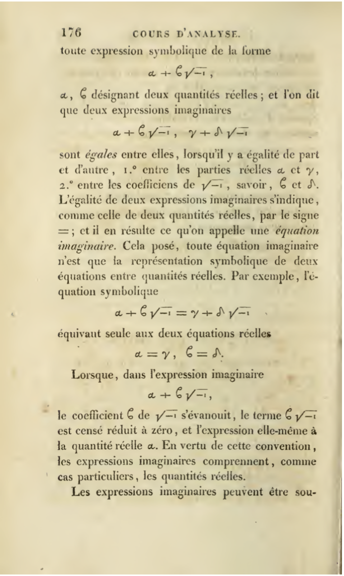
Complex numbers in the *Cours d'analyse* (1821)


176 COURS D'ANALYSE.

toute expression symbolique de la forme

$$\alpha + \beta\sqrt{-1},$$

α, β désignant deux quantités réelles ; et l'on dit que deux expressions imaginaires

$$\alpha + \beta\sqrt{-1}, \quad \gamma + \delta\sqrt{-1}$$

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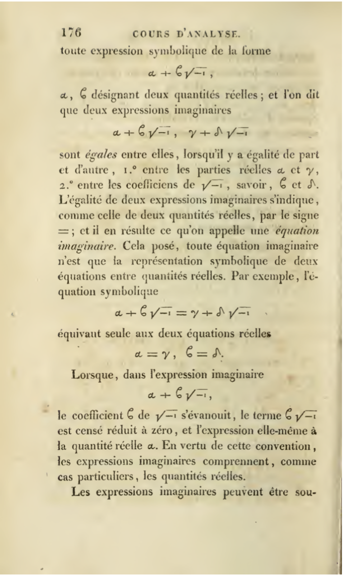
Complex numbers in the *Cours d'analyse* (1821)



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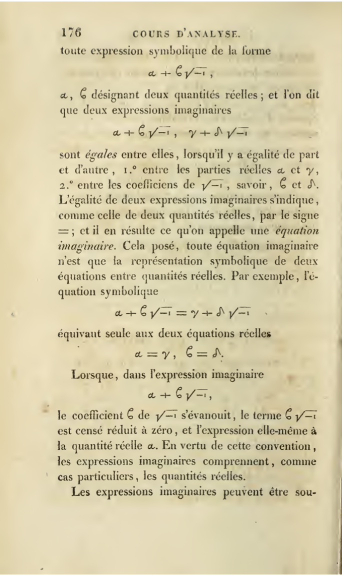

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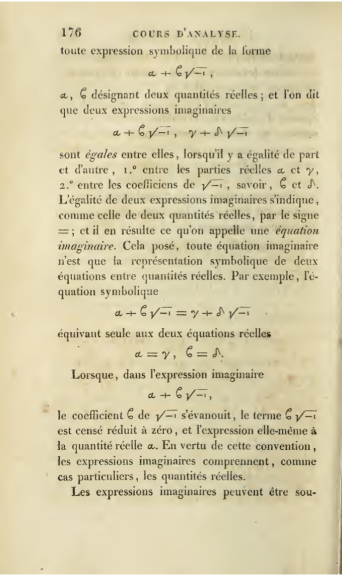

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Sought to extend ideas for real functions to the complex case, particularly those relating to power series and convergence

Cauchy's second 'Mémoire' (1825)

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(Gauss had found this in 1811, alongside a similar definition of a complex integral, but did not publish.)

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For the case where $f(x + y\sqrt{-1})$ becomes infinite at the point $x = a, y = b$, Cauchy considered the limit

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With a natural extension of this result for multiple and/or higher-order singularities, this became an ancestor of Cauchy's residue theorem — developed as part of Cauchy's calculus of residues in a paper of 1826 ('Sur un nouveau genre de calcul').

Taylor's Theorem for complex analytic functions

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1841: extension to negative powers — Laurent's Theorem.

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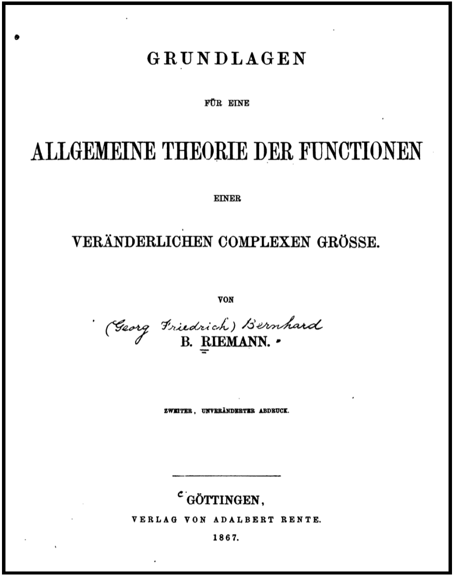
Point to note: Cauchy may be credited with many of the fundamental ideas of complex analysis, but this does not mean that they appeared fully-formed.

Riemann on complex analysis



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VON

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ZWEITER, UNVERÄNDERTER ABDRUCK.

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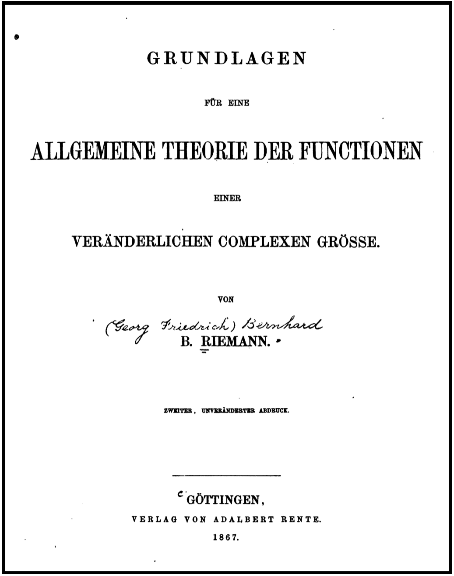
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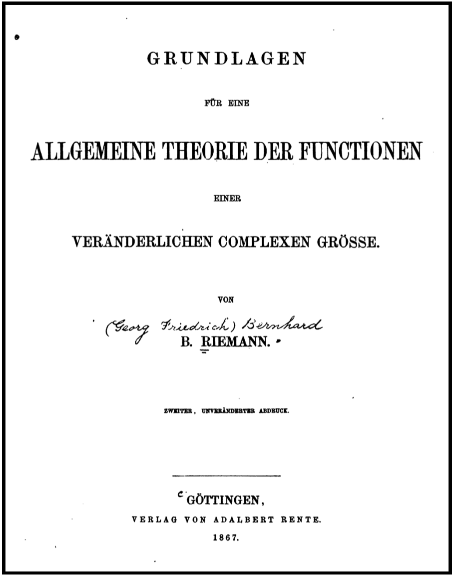
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That is: $\lim_{\delta \to 0} \frac{f(z+\delta)-f(z)}{\delta}$ exists

Riemann on complex analysis

— 4 —

so erhellt, dass er und zwar nur dann für je zwei Werthe von dx und dy denselben Werth haben wird, wenn

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Für die folgenden Betrachtungen beschränken wir die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes O nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. Wir wählen diese Einkleidung, bei der es unanstössig sein wird, von aufeinander liegenden Flächen zu reden, um die Möglichkeit offen zu lassen, dass der Ort des Punktes O über denselben Theil der Ebene sich mehrfach erstrecke; setzen jedoch für einen solchen Fall voraus, dass die auf einander liegenden Flächentheile nicht längs einer Linie zusammenhängen, so dass eine Umfaltung der Fläche, oder eine Spaltung in auf einander liegende Theile nicht vorkommt.

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Cauchy–Riemann equations now taken as fundamental to the theory

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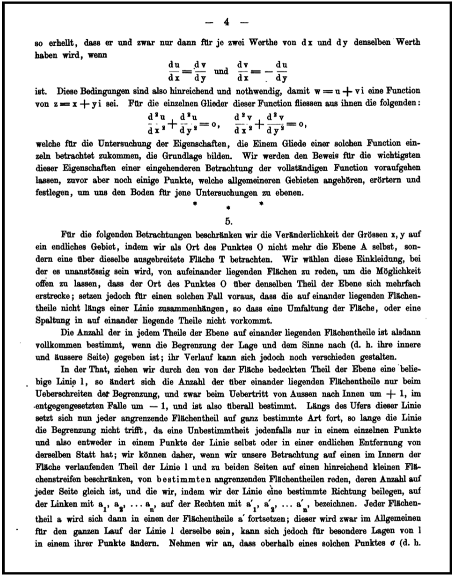

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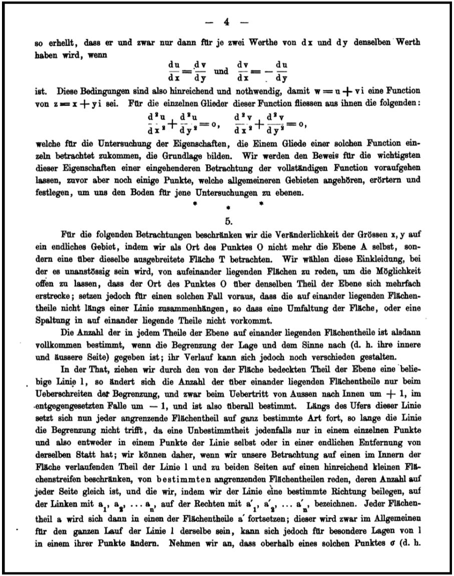
Other key concepts appear explicitly:

- harmonic functions;

Riemann on complex analysis



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Cauchy–Riemann equations now taken as fundamental to the theory

Other key concepts appear explicitly:

- harmonic functions;
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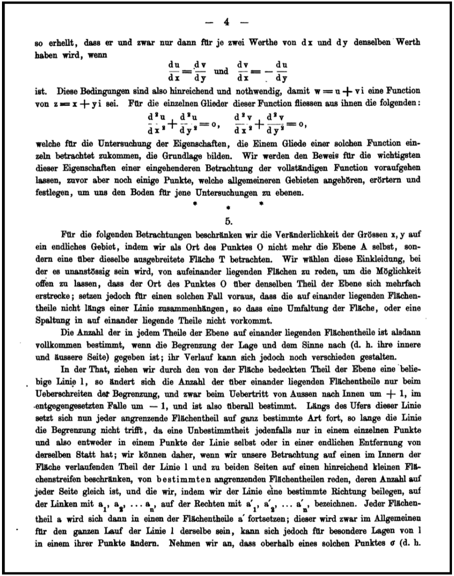
Riemann on complex analysis



— 4 —

so erhellt, dass er und zwar nur dann für je zwei Werthe von dx und dy denselben Werth haben wird, wenn

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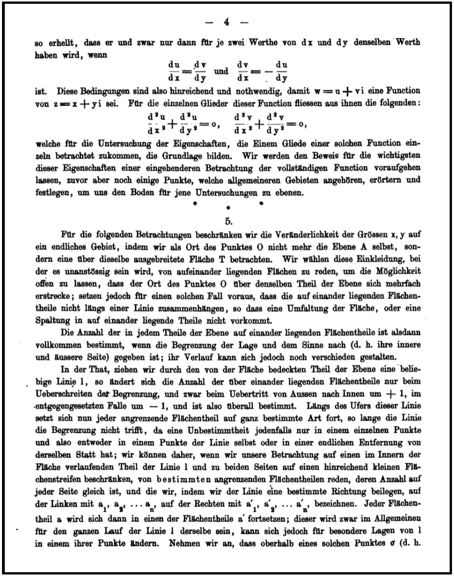

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