C4.3 Functional Analytic Methods for PDEs
Lecture 14

Luc Nguyen
luc.nguyen@maths

University of Oxford

MT 2021

In the last 3 lectures

- First and second existence theorems of weak solutions to linear elliptic equations.

This lecture

- Third existence theorem: Spectral theory.
- H^2 regularity of weak solutions to linear elliptic equations.

Spectra of elliptic operators

Theorem (Spectrum of an elliptic operator)

Suppose that Ω is a bounded Lipschitz domain. Suppose that $a, b, c \in L^\infty(\Omega)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Then there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{EBVP}$$

has a unique solution if and only if $\lambda \notin \Sigma$. Furthermore, if Σ is infinite then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ with

$$\lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$

The set Σ is called the real spectrum of the operator L.

Spectra of elliptic operators

Proof

- Let B be the bilinear form associated with L. Recall the energy estimate: There exists $\mu > 0$ depending on the L^∞ bounds for a, b, c and the ellipticity constant λ such that

$$\frac{\lambda}{2}\|u\|^2_{H^1(\Omega)} \leq B(u,u) + \mu\|u\|^2_{L^2(\Omega)}.$$

- If we define $L_\mu u = Lu + \mu u$ and let B_μ be the bilinear form associated with L_μ, then the right hand side above is exactly $B_\mu(u,u)$.
- So B_μ is coercive. By the Fredholm alternative, the operator $L_\mu : H^1_0(\Omega) \to H^{-1}(\Omega)$ is invertible. Denote its inverse by S_μ.

Spectra of elliptic operators

Proof

- Define an operator $K : L^2(\Omega) \to L^2(\Omega)$ by:

$$K : L^2(\Omega) \overset{J}{\hookrightarrow} H^{-1}(\Omega) \overset{S_\mu}{\to} H_0^1(\Omega) \overset{Id}{\hookrightarrow} L^2(\Omega).$$

  The last leg is compact by Rellich-Kondrachov's theorem, hence K is compact.
  (We also know that J is compact, but that is a harder statement.)
- Let Σ be the set of $\lambda \in \mathbb{R}$ such that (EBVP) is not always uniquely solvable. By the Fredholm alternative,

$$\begin{aligned} \lambda \in \Sigma &\Leftrightarrow (L - \lambda Id) \text{ is not injective} \\ &\Leftrightarrow (L_\mu - (\lambda + \mu) Id) \text{ is not injective} \\ &\Leftrightarrow I - (\lambda + \mu) K \text{ is not injective} \\ &\Leftrightarrow \lambda + \mu \neq 0 \text{ and } (\lambda + \mu)^{-1} \in \sigma_p(K). \end{aligned}$$

Spectra of elliptic operators

Proof

- ... $\lambda \in \Sigma$ if and only if $\lambda + \mu \neq 0$ and $(\lambda + \mu)^{-1} \in \sigma_p(K)$. The conclusion follows from a general result for spectra of compact operators, which we take for granted.

Theorem (Spectra of compact operators)

Let H be a Hilbert space of infinite dimension, $K : H \to H$ be a compact bounded linear operator and $\sigma(K)$ be its spectrum (i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda I - K$ is not invertible). Then

(i) 0 *belongs to* $\sigma(K)$.

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$, *i.e.* $\lambda I - K$ *has non-trivial kernel for* $\lambda \in \sigma(K) \setminus \{0\}$.

(iii) $\sigma(K) \setminus \{0\}$ *is either finite or an infinite sequence tending to* 0.

The question of regularity

In the rest of this course we consider regularity results for solutions to

$$Lu = -\partial_i(a_{ij}\partial_j u) + b_i \partial_i u + cu = f \text{ in a domain } \Omega$$

with $f \in L^2(\Omega)$.

- We want to keep in mind the following two motivating examples in $1d$:

$$-u'' = f \text{ in } (-1,1) \qquad (*)$$

  and

$$-(au')' = f \text{ in } (-1,1) \text{ where } a = \chi_{(-1,0)} + 2\chi_{(0,1)}. \qquad (**)$$

- For (*), u belongs to H^2.
- For (**), au' belongs to H^1. Typically this implies u' is discontinuous and hence $u \notin H^2$. Nevertheless u is continuous.

Interior H^2 regularity

Theorem (Interior H^2 regularity)

Suppose that $a \in C^1(\Omega)$, $b, c \in L^\infty(\Omega)$, *a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. *Suppose that* $f \in L^2(\Omega)$.
If $u \in H^1(\Omega)$ *satisfies* $Lu = f$ *in* Ω *in the weak sense then* $u \in H^2_{loc}(\Omega)$, *and for any open* ω *such that* $\bar{\omega} \subset \Omega$ *we have*

$$\|u\|_{H^2(\omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C *depends only on* $n, \Omega, \omega, a, b, c$.

Global H^2 regularity

Theorem (Global H^2 regularity)

Suppose that Ω is a bounded domain and $\partial\Omega$ is C^2 regular. Suppose that $a, b, c \in C^1(\bar{\Omega})$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ satisfies $Lu = f$ in Ω in the weak sense then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on n, Ω, a, b, c.

Remark: If $\partial\Omega$ is C^∞, $a, b, c \in C^\infty(\bar{\Omega})$, and $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

The case of $-\Delta$

To illustrate the idea, we focus in the case a is constant, $b \equiv 0$, $c \equiv 0$. The local H^2 regularity result is equivalent to:

Theorem (Interior H^2 regularity for $-\Delta$)

Suppose $f \in L^2(B_2)$ and $u \in H^1(B_2)$. If $-\Delta u = f$ in B_2 in the weak sense, then $u \in H^2(B_1)$ and

$$\|u\|_{H^2(B_1)} \leq C(\|f\|_{L^2(B_2)} + \|u\|_{H^1(B_2)})$$

where the constant C depends only on n.

The case of $-\Delta$

The start of the proof is the following simple but important lemma:

Lemma

Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. *Then*

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|\Delta u\|_{L^2(\mathbb{R}^n)}.$$

The proof is a computation using integration by parts:

$$\begin{aligned}\|\nabla^2 u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \partial_i\partial_j u \partial_i\partial_j u\,dx = -\int_{\mathbb{R}^n} \partial_j u \partial_j \partial_i^2 u\,dx\\ &= \int_{\mathbb{R}^n} \partial_j^2 u \partial_i^2 u\,dx = \|\Delta u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The case of $-\Delta$

The following lemma is a generalisation in the weak setting:

Lemma

Suppose that $f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$ and u has compact support. Suppose that $-\Delta u = f$ in $\mathbb{R}^n$ in the weak sense. Then $u \in H^2(\mathbb{R}^n)$ and

$$\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof of the lemma

- Take a family of mollifiers (ϱ_ε): Fix a non-negative function $\varrho \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^n} \varrho = 1$ and let $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.
- Set $u_\varepsilon = \varrho_\varepsilon * u$ and $f_\varepsilon = \varrho_\varepsilon * f$.
  Then $u_\varepsilon, f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$ and $f_\varepsilon \to f$ in $L^2(\mathbb{R}^n)$.

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ Fix $v \in C_c^\infty(\mathbb{R}^n)$ and consider $\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx$.
  - ⋆ Recall that, as $u \in H^1(\mathbb{R}^n)$, $\nabla u_\varepsilon = \varrho_\varepsilon * \nabla u$.
  - ⋆ Hence, by Fubini's theorem,

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx = \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\partial_{y_i} u(y)\, dy\Big] \partial_{x_i} v(x)\, dx$$
$$= \int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\partial_{x_i} v(x)\, dx\Big]\, dy.$$

  - ⋆ Integrating by parts in the inner integral we get

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\, dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y) \Big[\int_{\mathbb{R}^n} \partial_{x_i} \varrho_\varepsilon(x-y) v(x)\, dx\Big]\, dy.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ $\displaystyle\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = -\int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \partial_{x_i}\varrho_\varepsilon(x-y)v(x)\,dx\Big]\,dy.$
  - ⋆ Now observe that $\partial_{x_i}\varrho_\varepsilon(x-y) = -\partial_{y_i}\varrho_\varepsilon(x-y)$.
  - ⋆ We thus have, by Fubini's theorem again,

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = \int_{\mathbb{R}^n} \partial_{y_i} u(y)\Big[\int_{\mathbb{R}^n} \partial_{y_i}\varrho_\varepsilon(x-y)v(x)\,dx\Big]\,dy$$
$$= \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \partial_{y_i} u(y)\partial_{y_i}\varrho_\varepsilon(x-y)\,dy\Big]v(x)\,dx.$$

  - ⋆ As $-\Delta u = f$ in the weak sense, the inner integral is equal to $\displaystyle\int_{\mathbb{R}^n} f(y)\,\varrho_\varepsilon(x-y)\,dy$, which is $f_\varepsilon(x)$.
  - ⋆ We deduce that

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla v\,dx = \int_{\mathbb{R}^n} f_\varepsilon(x)v(x)\,dx.$$

The case of $-\Delta$

Proof of the lemma

- Claim: $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$.
  - ⋆ As v was picked arbitrarily in $C_c^\infty(\mathbb{R}^n)$, we have that $-\Delta u_\varepsilon = f_\varepsilon$ in $\mathbb{R}^n$ in the weak sense.
  - ⋆ As u_ε and f_ε are smooth, this equation also holds in the classical sense. (Check this!)
- Now, by the previous lemma, we have

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|\Delta u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|f_\varepsilon\|_{L^2(\mathbb{R}^n)}.$$

- Young's convolution inequality gives $\|f_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}\|\varrho_\varepsilon\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$, and so

$$\|\nabla^2 u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}.$$

- Therefore, along a subsequence, $(\nabla^2 u_\varepsilon)$ converges weakly to some $A \in L^2(\mathbb{R}^n; \mathbb{R}^{n\times n})$ with $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.

The case of $-\Delta$

Proof of the lemma

- Putting things together we have $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$, $\nabla^2 u_\varepsilon \rightharpoonup A$ in $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$.
- Claim: A is the weak second derivatives of u.
  Indeed, this follows by passing $\varepsilon \to 0$ in the identity

$$\int_{\mathbb{R}^n} u_\varepsilon \partial_i \partial_j v = \int_{\mathbb{R}^n} \partial_i \partial_j u_\varepsilon v \text{ for all } v \in C_c^\infty(\mathbb{R}^n).$$

- We have thus shown that $u \in H^2(\mathbb{R}^n)$ and $\|\nabla^2 u\|_{L^2(\mathbb{R}^n)} = \|A\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(B_2)}$.