Distribution Theory MT21

Problem Sheet 4

Problem 1. *This question provides a condition ensuring that the usual partial derivatives coincide with the distributional partial derivatives.* Prove Lemma 5.21 from the lecture notes: If the dimension $n \geq 2$ and $f \in \mathrm{C}^1(\mathbb{R}^n \setminus \{0\}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ has usual partial derivatives $\partial_j f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ for each direction $1 \leq j \leq n$, then also

$$\int_{\mathbb{R}^n} \partial_j f \varphi \, \mathrm{d}x = -\int_{\mathbb{R}^n} f \partial_j \varphi \, \mathrm{d}x$$

holds for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Give an example to show that it can fail for dimension $n = 1$. Show that for dimension $n = 1$ we instead have the following: If $f \in \mathrm{C}^1(\mathbb{R} \setminus \{0\}) \cap \mathrm{C}(\mathbb{R})$ and the usual derivative $f' \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, then

$$\int_{\mathbb{R}} f' \varphi \, \mathrm{d}x = -\int_{\mathbb{R}} f \varphi' \, \mathrm{d}x$$

holds for all $\varphi \in \mathscr{D}(\mathbb{R})$.

Problem 2. *Boundary values in the sense of distributions for holomorphic functions.*
(a) Prove that for each $n \in \mathbb{N}$,

$$\big(x + \mathrm{i}\varepsilon\big)^{-n} \to \big(x + \mathrm{i}0\big)^{-n} \quad \text{in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0,$$

where the distribution $\big(x + \mathrm{i}0\big)^{-n}$ was defined in Problem 2 on Sheet 3.

A holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H} = \big\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \big\}$ is said to be of slow growth if for each $R > 0$ there exist $m = m_R \in \mathbb{N}_0$ and $c = c_R \geq 0$ so

$$|f(z)| \leq \frac{c}{\mathrm{Im}(z)^m}$$

holds for all $z \in \mathbb{H}$ with $\big|\mathrm{Re}(z)\big| \leq R$ and $\mathrm{Im}(z) < 2$.
(b) Prove that if $f \colon \mathbb{H} \to \mathbb{C}$ is holomorphic of slow growth, then it has a boundary value in the sense of distributions:

$$\big\langle f(x + \mathrm{i}0), \varphi \big\rangle := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} f(x + \mathrm{i}\varepsilon) \varphi(x) \, \mathrm{d}x$$

exists for all $\varphi \in \mathscr{D}(\mathbb{R})$ and defines a distribution. [*Hint: Assume first that* $m = 0$ *above and let* $F \colon \mathbb{H} \to \mathbb{C}$ *be the holomorphic primitive with* $F(\mathrm{i}) = 0$. *Explain why* F *has a continuous extension to the closed upper half-plane* $\overline{\mathbb{H}}$ *and use this to conclude the proof in this special case.*]

Problem 3. *Distributions defined by finite parts.*
Recall from Sheet 2 that the distributional derivative of $\log|x|$ is the distribution $\mathrm{pv}\big(\frac{1}{x}\big)$ defined by the principal value integral

$$\big\langle \mathrm{pv}\big(\tfrac{1}{x}\big), \varphi \big\rangle := \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x)}{x} \, \mathrm{d}x \, , \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

In order to represent the higher order derivatives one can use finite parts: Let $n \in \mathbb{N}$ with $n > 1$. We then define $\mathrm{fp}\big(\frac{1}{x^n}\big)$ for each $\varphi \in \mathscr{D}(\mathbb{R})$ by the *finite part integral*

$$\big\langle \mathrm{fp}\big(\tfrac{1}{x^n}\big), \varphi \big\rangle := \int_{-\infty}^{\infty} \frac{\varphi(x) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} x^j - \frac{\varphi^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)}{x^n} \, \mathrm{d}x.$$

(a) Check that hereby $\mathrm{fp}\big(\frac{1}{x^n}\big)$ is a well-defined distribution on $\mathbb{R}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{pv}\big(\tfrac{1}{x}\big) = -\mathrm{fp}\big(\tfrac{1}{x^2}\big) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{fp}\big(\tfrac{1}{x^n}\big) = -n \mathrm{fp}\big(\tfrac{1}{x^{n+1}}\big)$$

for all $n > 1$. Is $\mathrm{fp}\big(\frac{1}{x^n}\big)$ homogeneous? (See Problem 4 on Sheet 2 for the definition of homogeneity.)
(b) Show that for $n > 1$ we have $x^n \mathrm{fp}\big(\frac{1}{x^n}\big) = 1$ and find the general solution to the equation $x^n u = 1$ in $\mathscr{D}'(\mathbb{R})$. What is the general solution to the equation $(x-a)^n v = 1$ in $\mathscr{D}'(\mathbb{R})$ when $a \in \mathbb{R} \setminus \{0\}$?
(c) *Optional.* Let $p(x) \in \mathbb{C}[x] \setminus \{0\}$ be a nontrivial polynomial. Describe the general solution $w \in \mathscr{D}'(\mathbb{R})$ to the equation

$$p(x) w = 1 \ \text{ in } \ \mathscr{D}'(\mathbb{R}).$$

Problem 4. A function $f \colon \mathbb{R} \to \mathbb{R}$ is *convex* if for all x_0, $x_1 \in \mathbb{R}$ and $\lambda \in (0,1)$ we have

$$f\big(\lambda x_1 + (1-\lambda) x_0\big) \leq \lambda f\big(x_1\big) + (1-\lambda) f\big(x_0\big). \tag{1}$$

A function $a \colon \mathbb{R} \to \mathbb{R}$ satisfying (1) with equality everywhere is called an *affine function*.
(a) Show that an affine function must have the form $a(x) = a_1 x + a_0$ for some constants a_0, $a_1 \in \mathbb{R}$. Show also that a function $f \colon \mathbb{R} \to \mathbb{R}$ is convex if and only if it for each compact interval $[\alpha, \beta] \subset \mathbb{R}$ has the property:

when a is affine and $f(x) \leq a(x)$ for $x \in \{\alpha, \beta\}$, then $f \leq a$ on $[\alpha, \beta]$

(b) Show that a convex function $f \colon \mathbb{R} \to \mathbb{R}$ satisfies the *3 slope inequality*:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

holds for all triples $x_1 < x_2 < x_3$. Deduce that a convex function must be continuous and that it is differentiable except for in at most countably many points.

Optional: Show that a convex function must be locally Lipschitz continuous: for each $r > 0$ there exists $L = L_r \geq 0$ so $|f(x) - f(y)| \leq L|x - y|$ holds for all $x, y \in [-r, r]$.
(c) Assume that $f \colon \mathbb{R} \to \mathbb{R}$ is twice differentiable. Show that f is convex if and only if

$$f''(x) \geq 0$$

holds for all $x \in \mathbb{R}$.
(d) Let $u \in \mathscr{D}'(\mathbb{R})$ and assume that $u'' \geq 0$ in $\mathscr{D}'(\mathbb{R})$. Show that u is represented by a convex function.

Problem 5. *More practice on order of distributions, positive distributions and mollification.*
In this question all functions and distributions are assumed real-valued.
(a) What does it mean to say that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ has order $m \in \mathbb{N}_0$? Show that a positive distribution $v \in \mathscr{D}'(\mathbb{R}^n)$ has order 0 and explain in what sense it is a measure.
If $u, v \in \mathscr{D}'(\mathbb{R}^n)$ are two distributions we write $u \leq v$ when $v - u \geq 0$, that is, when $v - u$ is a positive distribution. Show that if $u \leq v$, then u has order $m \in \mathbb{N}_0$ if and only if v has order $m \in \mathbb{N}_0$.
What can you say about a distribution $w \in \mathscr{D}'(\mathbb{R})$ that satisfies $w' \geq 0$?
(b) In each of the following cases find all the distributions $u \in \mathscr{D}'(\mathbb{R})$ that satisfies the given differential inequality:

(1) $u' \geq 1$

(2) $u' \geq u$

(3) $u'' \geq u$

(c) Denote $\operatorname{sgn}(t) = t/|t|$ for $t \in \mathbb{R} \setminus \{0\}$ and $\operatorname{sgn}(0) = 0$. Define for each $t > 0$ the function

$$T_t(y) = \sqrt{y^2 + t}, \quad y \in \mathbb{R}.$$

(i) Assume $f \in \mathrm{C}^\infty(\mathbb{R}^n)$. Calculate $T_t(f)\partial_j T_t(f)$ for each j and deduce the formula

$$\left|\nabla T_t(f)\right|^2 + T_t(f)\Delta T_t(f) = \left|\nabla f\right|^2 + f\Delta f.$$

Use this to conclude that

$$\Delta T_t(f) \geq \frac{f}{T_t(f)}\Delta f$$

holds on $\mathbb{R}^n$.
(ii) Assume $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $\Delta u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. Prove, for instance using mollifiers, that

$$\Delta|u| \geq \operatorname{sgn}(u)\Delta u \quad \text{in } \mathscr{D}'(\mathbb{R}^n).$$