C5.1 Solid Mechanics

Sheet 4 — MT21

Section A: Check Your Understanding

[The questions in this section are intended to be simple extensions of material covered in lectures, helping you to check you understand what has been covered. Sketch solutions will be provided; these problems will not be covered in intercollegiate classes.]

1. **Cylindrical polar coordinates** Consider cylindrical polar coordinates (r, θ, z) in which the regular Cartesian coordinates are given by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z.$$

Show that the unit vectors in the two systems satisfy

$$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)^{\mathrm{T}} = \mathbf{R}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^{\mathrm{T}}$$

for some orthogonal matrix $\mathbf{R}$ that you should specify.

Deduce that:

(i). The expression of the vector $\mathbf{a} = (a_1, a_2, a_3)^{\mathrm{T}}$ in cylindrical polar coordinates, denoted $\mathbf{a}_* = (a_r, a_\theta, a_z)^{\mathrm{T}}$, is given by

$$\mathbf{a}_* = \mathbf{R}\,\mathbf{a}.$$

(ii). The matrix of a tensor $\mathbf{T}$ in cylindrical polars (denoted $\mathbf{T}_*$) is given by

$$\mathbf{T}_* = \mathbf{R}\,\mathbf{T}\,\mathbf{R}^{\mathrm{T}}.$$

Solution: We have that $\mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ and $\mathbf{e}_r = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$ so that $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)^{\mathrm{T}} = \mathbf{R}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^{\mathrm{T}}$ where

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is clearly orthogonal.

(i). We can write $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. Substituting in the expressions for $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of the vectors in the new basis the expressions for $\mathbf{a}$ in cylindrical polars follows.

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(ii). The tensor $\mathbf{T} = T_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ (with α, β used to denote the directions in cylindrical coordinates). We can rewrite this as

$$\mathbf{T} = (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z) \otimes \mathbf{T}_*(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)^{\mathrm{T}} = [(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\mathbf{R}^{\mathrm{T}}] \otimes \mathbf{T}_*\mathbf{R}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^{\mathrm{T}}$$

and we immediately have $\mathbf{T} = \mathbf{R}^{\mathrm{T}}\,\mathbf{T}_*\,\mathbf{R}$ and the desired result follows from orthogonality of $\mathbf{R}$.

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2. Use the results from Question 1 to:

   (i). Show that the deformation gradient for a deformation $\mathbf{x} = r(R,\Theta,Z)\mathbf{e}_r + \theta(R,\Theta,Z)\mathbf{e}_\theta + z(R,\Theta,Z)\mathbf{e}_z$ in cylindrical polar coordinates is given by

$$\mathbf{F}_* = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{r}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{pmatrix}$$

   (ii). Calculate the deformation gradient in cylindrical polar coordinates, $\mathbf{F}_*$, for the deformation defined by

$$r = AR, \quad \theta = B\log R + C\Theta, \quad z = \frac{Z}{A^2C}$$

   and hence show that the invariants I_j $(j = 1,2,3)$ of $\mathbf{B}_*$ are constants.

Solution:

(i). With this notation the deformation gradient is given by

$$\mathbf{F}_* = \mathbf{R}\,\mathbf{F}\,\mathbf{R}_0$$

with $\mathbf{R}_0$ denoting the matrix in the reference configuration

and, noting that $\partial_R = \cos\Theta\partial_{X_1} + \sin\Theta\partial_{X_2}$ while

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. **Inverting the linear elastic constitutive relation** In lectures we showed that the most general constitutive law for a linearly elastic, isotropic solid is

$$T_{ij} = 2\mu E_{ij} + \lambda[\mathrm{tr}(\mathcal{E})]\delta_{ij}.$$

Invert this expression to give the stress tensor $[\mathcal{E}]_{ij} = E_{ij}$ in terms of the stress tensor $[\mathcal{T}]_{ij} = T_{ij}$.

Solution: Contracting the indices we immediately have that

$$\mathrm{tr}(\mathcal{T}) = (2\mu + 3\lambda)\,\mathrm{tr}(\mathcal{E})$$

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which we can use to rearrange the constitutive relation as

$$2\mu E_{ij} = T_{ij} - \lambda \mathrm{tr}(\mathcal{E})\delta_{ij} = T_{ij} - \frac{\lambda}{2\mu + 3\lambda}\mathrm{tr}(\mathcal{T})\delta_{ij}$$

so that

$$E_{ij} = \frac{1}{2\mu}T_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\mathrm{tr}(\mathcal{T})\delta_{ij}.$$

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Section B: Test Your Understanding

[The questions in this section are intended to go into further detail on material covered in lectures, helping you test and develop your understanding further. Your solutions to these problems should be submitted by the deadline for the Intercollegiate Classes, with solutions provided during those classes.]

4. **The incompressible spherical shell**

   Following the description in the lectures, we consider the symmetric deformation of an incompressible spherical shell. Assume that the material is characterized by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$. Let $\lambda = r/R$ and $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$.

   (i). Show that for a given internal pressure P, the deformation is determined by the solution of
   $$P = \int_{\alpha}^{\beta} \frac{h'(\lambda)}{1-\lambda^3}\, \mathrm{d}\lambda \tag{1}$$
   where $\alpha = \lambda_a = a/A$ and $\beta = \lambda_b = b/B$.

   (ii). Express β as a function of α.

   (iii). Integrate P as a function of α and plot the pressure-stretch curves $P - \alpha$ for a neo-Hookean and a Mooney-Rivlin strain-energy (take *e.g.* $A = 1, B = 2, \mu_1 = 1, \mu_2 = 0.03$). How is the behaviour of P different for these two functions for large values of α?

5. **The *thin* incompressible spherical shell** Let us explore the thin-shell limit of the previous problem.

   (i). To start, show that P viewed as a function of α satisfies the equation
   $$(\alpha - \alpha^{-2})\frac{\mathrm{d}P}{\mathrm{d}\alpha} = \frac{h'(\alpha)}{\alpha^2} - \frac{h'(\beta)}{\beta^2}. \tag{4}$$

   (ii). Now, if the shell is thin, we can write $B - A = \epsilon A$ where $\epsilon \ll 1$. Let $\lambda = \alpha(1 + O(\epsilon))$ and show that
   $$P = \epsilon \frac{h'(\lambda)}{\lambda^2} \tag{5}$$

   (iii). Let T be the surface tension, a force per unit current length along the surface, that is $(b - a)T_{\theta\theta}$. Show that
   $$T = \epsilon A \frac{h'(\lambda)}{2\lambda}. \tag{6}$$

   (iv). Show how the two last equalities are related to the Young-Laplace law for a spherical membrane. Is this a universal result (independent of the particular choice of the strain-energy)?

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6. **The Rivlin square**

An equibiaxial tension consists in pulling a square sample with equal tension by the four edges. Viewed as a three-dimensional material, it consists in applying to a cuboid equal distributed tensile normal Cauchy stress $T > 0$ on two pairs of opposite faces, while leaving the remaining two faces stress-free. It is assumed that the cuboid remains a cuboid during the deformation. Consider an incompressible Mooney–Rivlin material with strain-energy density function of the form

$$W = \frac{1}{2}\mu \left[\left(\frac{1}{2} + \alpha \right) (I_1 - 3) + \left(\frac{1}{2} - \alpha \right) (I_2 - 3) \right] \tag{9}$$

(i). From the Cauchy stress tensor and the deformation gradient, define the principal stresses (t_1, t_2, t_3) and the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ and write down the constitutive relationship between them [take the direction $\mathbf{e}_3$ to be normal to the stress-free faces]. Also write down the incompressibility condition in terms of the principal stretches.

(ii). The Baker-Ericksen inequalities state that $(\lambda_i - \lambda_j)(t_i - t_j) > 0$ for $\lambda_i \neq \lambda_j$. Show that these inequalities imply that $-1/2 \leq \alpha \leq 1/2$ and $\mu > 0$.

(iii). Define the boundary conditions and compute the applied load T as a function of the stretches only.

(iv). Derive a relationship between λ_1 and λ_2 independent of T.

(v). Show that there is always a trivial solution for which $\lambda_1 = \lambda_2$ and that this solution is the only solution in the neo-Hookean case ($\alpha = 1/2$).

(vi). Show that there is only one possible homogeneous deformation for the Mooney-Rivlin material in equibiaxial tension and that T is a strictly increasing function of λ_1.

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Section C: Stretch Your Understanding

[The questions in this section are entirely optional; they are intended to go beyond the material covered in lectures, but should nevertheless help to consolidate your understanding. Solutions will be provided; these questions may be discussed in class ***if time allows****.]*

7. **The compressible spherical shell** Consider the symmetric deformation of a compressible spherical shell

$$\mathbf{x} = f(R)\mathbf{X}. \tag{10}$$

Assume that the material is characterized by a strain-energy density $W = W(\lambda_1, \lambda_2, \lambda_3)$.

(i). Find a second-order equation for $f(R)$ with coefficients functions of W and its derivatives with respect to λ_1, λ_2.

(ii). Give the explicit relationship between λ_1, λ_2 and $f(R)$.

(iii). Write explicitly (only as a function of R and $f(R)$) this equation for

$$W = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3). \tag{11}$$

(iv). Can you solve this equation? Analytically? Numerically? What would the boundary conditions be?

Solution:

(i). We denote the derivative with respect to R with a prime, $\mathrm{d}f/\mathrm{d}R = f'(R)$. The deformation gradient can be determined similarly to problem 8.2:

$$\mathbf{F} = \mathrm{diag}\,(f + Rf', f, f) = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$$

In this case, the nominal stress S (transpose of first Piola-Kirchhoff stress) stress is derived from the strain-energy density W as $S_{RR} = W_1$, $S_{\theta\theta} = W_2$ and $S_{\phi\phi} = W_3$ where $\partial W/\partial\lambda_i = W_i$. The linear momentum balance in the initial reference configuration reads $\mathrm{Div}\,\mathbf{S} = 0$, which is $S'_{RR}(R) + 2\left(S_{RR} - S_{\theta\theta}\right)/R = 0$. Evaluating this expression, we get

$$\lambda_1' W_{11} + \lambda_2' W_{12} + \lambda_3' W_{13} + \frac{2\left(W_1 - W_2\right)}{R} = 0$$

Substituting λ_1, λ_2 and λ_3, we get

$$\frac{2}{R}\left(W_1 - W_2\right) + 2f'W_{12} + \left(2f' + Rf''\right)W_{11} = 0$$

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(ii). From above we immediately obtain the explicit relations $\lambda_1 = f + Ff'$, and $\lambda_2 = f$

(iii). With the Mooney-Rivlin strain energy density, this equation becomes

$$\left(\mu_1 + 2\mu_2 f^2\right)\left(Rf'' + 4f'\right) + 2\mu_2 Rff'^2 = 0$$

(iv). This needs numerical solution. A typical boundary value problem for a spherical shell is $\mathbf{T}\left(-\mathbf{e}_r\right) = P\mathbf{e}_r$ at the inner wall and $\mathbf{T}\mathbf{e}_r = 0\mathbf{e}_r$ at the outer wall where $\mathbf{T}$ is the Cauchy stress. Transforming these two equations into the initial reference configuration by application of Nanson's formula, we have $\mathbf{S}^T\mathbf{E}_R = -Pf^2\mathbf{e}_r$ at $R = A$ and $\mathbf{S}^T\mathbf{E}_R = 0\mathbf{e}_r$ at $R = B$.

8. **Navier equations**.

For an isotropic linearly elastic material, the constitutive relationship is

$$\mathbf{T} = 2\mu\mathbf{e} + \lambda \mathrm{Tr}(\mathbf{e})\mathbf{1}. \tag{12}$$

where λ and μ are the classical Lamé parameters. Derive the static Navier equations for the displacements $\mathbf{u}$.

(i). Write the general equilibrium static equations for a compressible hyperelastic solid in the absence of body forces in the reference configuration.

(ii). Show that the positive definiteness of the strain energy for a linearly elastic, isotropic material, $\mathrm{C}_{\mathrm{iso}}$, implies both $2\mu + 3\lambda > 0$ and $\mu > 0$.

(iii). Let $\mathbf{u} \in C^4$ be a solution of the Navier equations. Show that both Div $\mathbf{u}$ and Curl $\mathbf{u}$ are harmonic functions, that is

$$\nabla^2 \text{ Div } \mathbf{u} = 0, \tag{13}$$

$$\nabla^2 \text{ Curl } \mathbf{u} = 0. \tag{14}$$

Furthermore, use these identities to prove that $\mathbf{u}$ is a biharmonic function, that is $\nabla^2\nabla^2\mathbf{u} = 0$.

Hint: You may use without proof the following identities:

$$\nabla^2\mathbf{u} = \text{Grad Div } \mathbf{u} - \text{Curl Curl } \mathbf{u}, \tag{15}$$

$$\text{Div Curl } \mathbf{u} = 0. \tag{16}$$

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Solution: The static Navier equation is simply $\nabla \cdot \mathbf{T} = 0$ (using symmetry of $\mathbf{T}$).

(i). We have that

$$T_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

so that $\partial_i T_{ij} = 0$ immediately becomes:

$$0 = (\lambda + \mu)\frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

or

$$0 = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u}. \tag{17}$$

(ii). A simple calculation shows that the stress tensor $\mathbf{T}$ in (12) can be derived as $T_{ij} = \partial \mathrm{C}_{\mathrm{iso}}/\partial e_{ij}$ where

$$\mathrm{C}_{\mathrm{iso}} = \frac{\lambda}{2}(e_{kk})^2 + \mu e_{ij}e_{ij}.$$

To derive conditions for positive definiteness, consider $e'_{ij} = e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}$; e'_{ij} is then trace-free and

$$\mathrm{C}_{\mathrm{iso}} = \frac{1}{2}(\lambda + \tfrac{2}{3}\mu)(e_{kk})^2 + \mu e'_{ij}e'_{ij}.$$

If $\lambda + \frac{2}{3}\mu > 0$ and $\mu > 0$ then this has a global minimum of 0 when $e_{ij} = 0$. Conversely, considering the cases of pure shear and pure dilation given in lectures shows that $\mathrm{C}_{\mathrm{iso}} < 0$ if either of these combinations is negative, so that the zero strain state is no longer the minimum.

(iii). Taking the divergence of (17) we have

$$0 = (\lambda + 2\mu)\nabla^2(\nabla \cdot \mathbf{u})$$

which clearly gives the first desired result.

Taking the curl of (17) we have:

$$0 = (\lambda + \mu)\nabla \wedge \nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2(\nabla \wedge \mathbf{u}),$$

which (since the first term vanishes) gives the second result.

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