C4.3 Functional Analytic Methods for PDEs
Lecture 15

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In the last lecture

- H^2 regularity of weak solutions to linear elliptic equations.

This lecture

- H^2 regularity of weak solutions to linear elliptic equations.
- Continuity of weak solutions to linear elliptic equations.

A priori H^2 estimates in the general case

- We prove for $a_{ij} = \delta_{ij}$ that if $u \in H^1(\mathbb{R}^n)$ satisfies $-\partial_i(a_{ij}\partial_j u) = f$ on $\mathbb{R}^n$ with $f \in L^2(\mathbb{R}^n)$, then $u \in H^2(\mathbb{R}^n)$.
- We now turn to the case where a is variable. To better convey central ideas, we will focus in the rest of this course to a priori estimates: We assume that the solution has the right regularity and will be concerned with establishing quantitative estimates.
- More precisely, we suppose that u belongs to $\underline{H^2(\mathbb{R}^n)}$ and is a weak solution to $Lu = f$ in $\mathbb{R}^n$, and would like to bound $\|u\|_{H^2(\mathbb{R}^n)}$ in terms of the bounds for the coefficients of L, $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|u\|_{H^1(\mathbb{R}^n)}$.
- For simplicity, we will assume that $b \equiv 0$ and $c \equiv 0$. You should check that the methods we use work in the general case.

Method of freezing coefficients

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ *and* $L = -\partial_i(a_{ij}\partial_j)$.
There exist $0 < \delta_0 \ll 1$ *and* $C > 0$ *such that if* $\|a_{ij} - \delta_{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \delta_0$ *and if* $u \in H^2(\mathbb{R}^n)$ *and satisfies* $Lu = f$ *in* $\mathbb{R}^n$ *in the weak sense, then*

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Claim: u satisfies

$$-\Delta u = f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u =: \tilde{f},$$

that is, for all $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u\Big] v \, dx.$$

Method of freezing coefficients

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij}) \partial_i \partial_j u + \partial_i a_{ij} \partial_j u \Big] v \, dx.$$

  - ⋆ We note that $(a_{ij} - \delta_{ij}) v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\begin{aligned}
\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij}) \partial_i \partial_j u v \, dx &= - \int_{\mathbb{R}^n} \partial_j u \partial_i [(a_{ij} - \delta_{ij}) v] \, dx \\
&= - \int_{\mathbb{R}^n} \partial_j u [(a_{ij} - \delta_{ij}) \partial_i v + \partial_i a_{ij} v] \, dx \\
&= \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^n} a_{ij} \partial_j u \partial_i v \, dx \\
&\qquad - \int_{\mathbb{R}^n} \partial_i a_{ij} v \, dx.
\end{aligned}$$

Method of freezing coefficients

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i \partial_j u + \partial_i a_{ij} \partial_j u \Big] v \, dx.$$

  - ⋆ $$\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij})\partial_i \partial_j u v \, dx = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^n} a_{ij} \partial_j u \partial_i v \, dx - \int_{\mathbb{R}^n} \partial_i a_{ij} v \, dx.$$

  - ⋆ As $Lu = f$, we have

$$\int_{\mathbb{R}^n} a_{ij} \partial_j u \partial_i v \, dx = \int_{\mathbb{R}^n} f \, v \, dx.$$

  - ⋆ Putting the two identities together, we obtain the claim.

Method of freezing coefficients

Proof

- We have proved the claim that $-\Delta u = \tilde{f} = f + (a_{ij} - \delta_{ij})\partial_i \partial_j u + \partial_i a_{ij} \partial_j u$.
- By the lemma on the H^2 regularity for $-\Delta$, we have a constant C such that

$$\begin{aligned}\|\nabla^2 u\|_{L^2} &\leq C\|\tilde{f}\|_{L^2} \\ &\leq C\Big[\|f\|_{L^2} + \|a_{ij} - \delta_{ij}\|_{L^\infty}\|\nabla^2 u\|_{L^2(\Omega)} \\ &\qquad + \|\partial_i a_{ij}\|_{L^\infty}\|\nabla u\|_{L^2}\Big].\end{aligned}$$

- It is readily seen that if $C\|a_{ij} - \delta_{ij}\|_{L^\infty} < 1$, then the second term on the right hand side can be absorbed back to the left hand side, giving the conclusion:

$$\|\nabla^2 u\|_{L^2} \leq C'\Big[\|f\|_{L^2} + \|\nabla u\|_{L^2}\Big].$$

Method of differentiating the equation

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ *and* $L = -\partial_i(a_{ij}\partial_j)$.
There exists $C > 0$ *such that if* $u \in H^2(\mathbb{R}^n)$ *and satisfies* $Lu = f$ *in* $\mathbb{R}^n$ *in the weak sense, then*

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Let $w = \partial_k u \in H^1(\mathbb{R}^n)$. We would like to bound $\|w\|_{H^1}$.
- Claim: w satisfies

$$Lw = \partial_i h_i \text{ where } h_i = \partial_k a_{ij}\partial_j u + f\,\delta_{ik},$$

that is, for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\,dx.$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\,dx.$$

  ⋆ Note that $a_{ij}\partial_i v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = \int_{\mathbb{R}^n} \partial_k \partial_j u\,(a_{ij}\partial_i v)\,dx = -\int_{\mathbb{R}^n} \partial_j u\,\partial_k(a_{ij}\partial_i v)\,dx$$
$$= -\int_{\mathbb{R}^n} a_{ij}\partial_j u\,\partial_k \partial_i v\,dx - \int_{\mathbb{R}^n} \partial_j u\,\partial_k a_{ij}\partial_i v\,dx$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\,dx.$$

  - ⋆ $\displaystyle\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} a_{ij}\partial_j u\,\partial_k\partial_i v\,dx - \int_{\mathbb{R}^n} \partial_j u\,\partial_k a_{ij}\partial_i v\,dx.$
  - ⋆ On the other hand, using $\partial_k v$ as a test function for $Lu = f$, we have
$$\int_{\mathbb{R}^n} a_{ij}\partial_j u\,\partial_i\partial_k v\,dx = \int_{\mathbb{R}^n} f\,\partial_k v\,dx.$$
  - ⋆ Putting the two identities together we get the claim.

Method of differentiating the equation

Proof

- We have thus shown that $Lw = \partial_i h_i$ with $h_i = \partial_k a_{ij} \partial_j u + f\,\delta_{ik}$.
- Using w as a test function for this equation, we get

$$\int_{\mathbb{R}^n} a_{ij} \partial_j w \partial_i w \, dx = -\int_{\mathbb{R}^n} h_i \partial_i w \, dx.$$

- Using ellipticity on the left side and Cauchy-Schwarz' inequality on the right side we get

$$\lambda \|\nabla w\|_{L^2}^2 \leq \|h\|_{L^2} \|\nabla w\|_{L^2} \leq \frac{\lambda}{2} \|\nabla w\|_{L^2}^2 + \frac{1}{2\lambda} \|h\|_{L^2}^2.$$

- We thus have

$$\|\nabla w\|_{L^2} \leq C \|h\|_{L^2} \leq C \Big[\|f\|_{L^2} + \|\nabla u\|_{L^2} \Big].$$

Recalling that $w = \partial_k u$, we're done.

Example

- Recall the example of the equation $-(au')' = f$ in $(-1, 1)$ with $a = \chi_{(-1,0)} + 2\chi_{(0,1)}$.
- If $f \in L^q$, then $au' \in W^{1,q}$ and so u' is presumably discontinuous.
- Nevertheless as u' exists by assumption, u is continuous.
- In higher dimension, the existence of ∇u (in L^2) doesn't ensure continuity of u. Nevertheless, a major result due to De Giorgi, Moser and Nash around late 50s asserts that u is indeed continuous!

De Giorgi-Moser-Nash's theorem

Theorem (De Giorgi-Moser-Nash's theorem)

Suppose that $a, b, c \in L^\infty(\Omega)$, *a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. *If* $u \in H^1(\Omega)$ *satisfies* $Lu = f$ *in* Ω *in the weak sense for some* $f \in L^q(\Omega)$ *with* $q > \frac{n}{2}$, *then u is locally Hölder continuous , and for any open* ω *such that* $\bar{\omega} \subset \Omega$ *we have*

$$\|u\|_{C^{0,\alpha}(\omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on $n, \Omega, \omega, a, b, c$, *and the Hölder exponent* α *depends only on* n, Ω, ω, a.

A digression

We make some observations:

- In De Giorgi-Moser-Nash's theorem, no continuity is assumed on the coefficients a_{ij}.
- If a_{ij} is continuous, one can imagine using the method of freezing coefficients to reduce to the case a_{ij} is constant. Hence the model equation is $-\Delta u = f$.
- In $1d$, we have $-u'' = f$. If $f \in L^q$, we then have that $u \in W^{2,q}_{loc}$.
- It turns out that, in any dimension, if $-\Delta u = f$ and $f \in L^q$, then $u \in W^{2,q}_{loc}$.
  In particular, when $n/2 < q < n$, by the embedding $W^{2,q}_{loc} \hookrightarrow W^{1,\frac{qn}{n-q}}_{loc} \hookrightarrow C^{0,2-\frac{n}{q}}_{loc}$, we have u is Hölder continuous.

Global a priori L^∞ estimate

To illustrate the method, we will assume for simplicity that $b \equiv 0$ and $c \equiv 0$. We will focus on a priori L^∞ estimates, i.e. we assume that the solution $u \in L^\infty$ and try to establish estimates for $\|u\|_{L^\infty}$.

- We assume in addition for now a boundary condition: $u = 0$ on ∂B_1.

Theorem (Global a priori L^∞ estimates)

Suppose that $a \in L^\infty(B_1)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H_0^1(B_1) \cap L^\infty(B_1)$ satisfies $Lu = f$ in B_1 in the weak sense and $f \in L^q(B_1)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)})$$

where the constant C depends only on n, q, a.

Truncations and powers of H^1 functions

Lemma

Suppose that $u \in H_0^1(B_1) \cap L^\infty(B_1)$. Then, for $p \geq 1$ and $k \geq 0$, one has $(u_+ + k)^p - k^p \in H_0^1(B_1)$.

Proof

- As $u \in L^\infty(B_1)$, we can suppose $|u| \leq M$ a.e. in B_1.
- By Sheet 3, $u_+ \in H^1(B_1)$.
- Select a function $g \in C^1(\mathbb{R})$ such that $g(t) = (t_+ + k)^p - k^p$ for $t \leq M$, and $g(t) = (M + k + 1)^p - k^p$ for $t \geq M + 1$. Note that $(u_+ + k)^p - k^p = g(u)$.
- Then $|g(t)| + |g'(t)| \leq C$ on $\mathbb{R}$.
- By the chain rule (Sheet 2), $g(u)$ has weak derivatives $\nabla g(u) = g'(u)\nabla u \in L^2(B_1)$. Hence $g(u) \in H^1(B_1)$.

Truncations and powers of H^1 functions

Proof

- $g(u) \in H^1(B_1)$.
- We next show that $g(u) \in H^1_0(B_1)$.
  Approximate u by $(u_m) \in C_c^\infty(B_1)$. The argument above shows that $g(u_m) \in H^1(B_1)$.
  As $g(u_m)$ is continuous, we have that the its trace on ∂B_1 is zero, hence $g(u_m) \in H^1_0(B_1)$.
- We have, by Lebesgue's dominated convergence theorem

$$\int_{B_1} |g(u_m) - g(u)|^2 \, dx \to 0.$$

So $g(u_m) \to g(u)$ in L^2.

Proof

- Next, we have

$$\begin{aligned}\int_B |\nabla g(u_m) - \nabla g(u)|^2\,dx &= \int_B |g'(u_m)\nabla u_m - g'(u)\nabla u|^2\,dx\\ &\leq \int_B |g'(u_m) - g'(u)|^2 |\nabla u|^2\,dx\\ &\quad + \int_B |g'(u_m)|^2 |\nabla u_m - \nabla u|^2\,dx \to 0,\end{aligned}$$

  where we use Lebesgue's dominated convergence theorem to treat the first integral and the convergence of ∇u_m to ∇u in L^2 to treat the second integral.
  Hence $\nabla g(u_m) \to \nabla g(u)$ in L^2.
- We have thus shown that $g(u_m) \in H_0^1(B)$ and $g(u_m) \to g(u)$ in $H^1(B)$. The conclusion follows.

Global a priori L^∞ estimates

We now prove the statement that if $u \in H_0^1(B_1) \cap L^\infty(B_1)$ is such that $Lu = f$ in B_1 with $f \in L^q(B_1)$ for some $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)}).$$

- We use Moser iteration method. We write $B = B_1$ and fix some $k > 0$, $p \geq 1$.
- Let $w = u_+ + k$ and we use $v = w^p - k^p$ as test function. This is possible because we just proved that $v \in H_0^1(B_1)$.
  We have

$$\begin{aligned}
\int_B f\, v\, dx &= \int_B a_{ij}\partial_j u \partial_i v\, dx \\
&= \int_B p w^{p-1} a_{ij}\partial_j u \partial_i u_+\, dx \\
&\overset{ellipticity}{\geq} \lambda p \int_B w^{p-1} |\nabla u_+|^2\, dx.
\end{aligned}$$

Global a priori L^∞ estimate

Proof

- We thus have
$$\int_B |\nabla w^{\frac{p+1}{2}}|^2\,dx \leq Cp\int_B |f|\,|v|\,dx \leq Cp\int_B |f| w^p\,dx.$$
- By Friedrichs' inequality, this gives
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{H^1}^2 \leq Cp\int_B |f| w^p\,dx.$$
- By Gagliardo-Nirenberg-Sobolev's inequality, this implies that
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \leq Cp\int_B |f|\,w^p\,dx.$$
- We thus have
$$\|w^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \leq Cp\int_B (\frac{|f|}{k}+1)\,w^{p+1}\,dx.$$

Global a priori L^∞ estimate

Proof

- $\|w^{\frac{p+1}{2}}\|^2_{L^{\frac{2n}{n-2}}} \leq Cp \int_B (\frac{|f|}{k} + 1)\, w^{p+1}\, dx.$
- Using Hölder's inequality, we then arrive at

$$\|w^{p+1}\|_{L^{\frac{n}{n-2}}} \leq Cp(\|\frac{|f|}{k}\|_{L^q} + 1)\|w^{p+1}\|_{L^{q'}}.$$

- We now choose k to be any number larger than $\|f\|_{L^q}$ and obtain from the above that

$$\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq Cp\|w\|^{p+1}_{L^{q'(p+1)}}.$$

Recalling that $q > n/2$, we have $q' < \frac{n}{n-2}$. Thus the above inequality is self-improving: If w has a bound in $L^{q'(p+1)}$, then it has a bound in $L^{\frac{n(p+1)}{n-2}}$.

Global a priori L^∞ estimate

Proof

- $\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq C(p+1)\|w\|^{p+1}_{L^{q'(p+1)}}$.
- Now let $\chi = \frac{n}{(n-2)q'} > 1$ and $t_m = \gamma\chi^m$ for some $\gamma > 2q'$, then the above gives

$$\|w\|_{L^{t_{m+1}}} \leq (Ct_m)^{\frac{q'}{t_m}}\|w\|_{L^{t_m}}$$
$$= (C\gamma)^{q'\gamma^{-1}\chi^{-m}}\chi^{q'\gamma^{-1}m\chi^{-m}}\|w\|_{L^{t_m}}.$$

Hence by induction,

$$\|w\|_{L^{t_{m+1}}} \leq (C\gamma)^{q'\gamma^{-1}\sum_m \chi^{-m}}\chi^{q'\gamma^{-1}\sum_m m\chi^{-m}}\|w\|_{L^\gamma} \leq C\|w\|_{L^\gamma}.$$

- Sending $m \to \infty$, we obtain

$$\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma} \text{ provided } \gamma > 2q'.$$

Global a priori L^∞ estimate

Proof

- $\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma}$ when $\gamma > 2q'$.
- We now reduce from L^γ to L^2:

$$\|w\|_{L^\infty} \leq C\Big\{\int_B |w|^\gamma\,dx\Big\}^{1/\gamma} \leq C\|w\|_{L^\infty}^{1-\frac{2}{\gamma}}\Big\{\int_B |w|^2\,dx\Big\}^{1/\gamma}.$$

  This gives

$$\|w\|_{L^\infty} \leq C\|w\|_{L^2}.$$

- Recalling that $w = u_+ + k$ and k can be any positive constant larger than $\|f\|_{L^q}$, we have thus shown that

$$\|u_+\|_{L^\infty} \leq C(\|u\|_{L^2} + \|f\|_{L^q})$$

- Applying the same argument to u_-, we get the corresponding bound for u_- and conclude the proof.