C4.3 Functional Analytic Methods for PDEs
Lecture 16

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In the last lecture

- De Giorgi-Moser-Nash's theorem on the continuity of weak solutions to linear elliptic equations.
- A priori L^∞ estimates.

This lecture

- A priori L^∞ estimates.
- Other topics.

Global a priori L^∞ estimate

Theorem (Global a priori L^∞ estimates)

Suppose that $a, b, c \in L^\infty(B_1)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H_0^1(B_1) \cap L^\infty(B_1)$ satisfies $Lu = f$ in B_1 in the weak sense and $f \in L^q(B_1)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)})$$

where the constant C depends only on n, q, a, b, c.

Remark

When L is injective, the term $\|u\|_{L^2(B_1)}$ on the right hand side can be dropped yielding the estimate:

$$\|u\|_{L^\infty(B_1)} \leq C\|f\|_{L^q(B_1)}.$$

Energy estimate with L^q right hand side

The remark is a consequence of:

Theorem

Suppose that $a, b, c \in L^\infty(B_1)$*, a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$*. Suppose that the only solution in* $H_0^1(B_1)$ *to* $Lu = 0$ *is the trivial solution. Then, for every* $u \in H_0^1(B_1)$ *and* $f \in L^q(B_1)$ *with* $q \geq \frac{2n}{n+2}$ *satisfying* $Lu = f$ *in* B_1*, there holds*

$$\|u\|_{H^1(B_1)} \leq C\|f\|_{L^q(B_1)}$$

where the constant C *depends only on* n, q, a, b, c*.*

Proof

- When $q = 2$, the result is a consequence of the Fredholm alternative and the inverse mapping theorem.

Energy estimate with L^q right hand side

Proof

- Let us consider first the case that $b \equiv 0$ and $c \equiv 0$.
  - ⋆ In this case, by using u as a test function, we have

$$\lambda \|\nabla u\|_{L^2}^2 \leq \int_{B_1} a_{ij} \partial_j u \partial_i u \, dx = \int_B f u \, dx \leq \|f\|_{L^q} \|u\|_{L^{q'}}.$$

  - ⋆ By Friedrichs' inequality, we have $\|u\|_{H^1} \leq C \|\nabla u\|_{L^2}$. As $q \geq \frac{2n}{n+2}$, $q' \leq \frac{2n}{n-2}$. Hence, by Gagliardo-Nirenberg-Sobolev's inequality, $\|u\|_{L^{q'}} \leq C\|u\|_{H^1}$.
  - ⋆ Therefore

$$\|u\|_{H^1}^2 \leq C \|\nabla u\|_{L^2}^2 \leq C \|f\|_{L^q} \|u\|_{L^{q'}} \leq C \|f\|_{L^q} \|u\|_{H^1},$$

from which we get $\|u\|_{H^1} \leq C \|f\|_{L^q}$, as desired.

Energy estimate with L^q right hand side

Proof

- Let us now consider the general case. By using u as a test function, we have

$$B(u,u) = \int_{B_1} fu\,dx \leq \|f\|_{L^q}\|u\|_{L^{q'}},$$

where B is the bilinear form associated with L.

- The right hand side is treated as before and is bounded from above by $C\|f\|_{L^q}\|u\|_{H^1}$. For the left hand side, we use Friedrichs' inequality together with energy estimates:

$$B(u,u) + C\|u\|_{L^2}^2 \geq \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 \geq \frac{1}{C}\|u\|_{H^1}^2.$$

We thus have

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2.$$

Energy estimate with L^q right hand side

Proof

- $\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2$.
- By Cauchy-Schwarz' inequality, we then have

$$\|u\|_{H^1}^2 \leq \frac{1}{2}\|u\|_{H^1}^2 + C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2,$$

  and so

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2.$$

- In other words,

$$\|u\|_{H^1} \leq C\|f\|_{L^q} + C\|u\|_{L^2}. \qquad (*)$$

- To conclude, we show that

$$\|u\|_{L^2} \leq C\|f\|_{L^q}. \qquad (**)$$

  More precisely, we show that "(*) + injectivity of L $\Rightarrow$ (**)".

Energy estimate with L^q right hand side

Proof

- Suppose by contradiction that there exists sequence $u_m \in H_0^1(B_1)$, $f_m \in L^q(B_1)$ such that $Lu_m = f_m$ but

$$\|u_m\|_{L^2} > m\|f_m\|_{L^q}.$$

  Replacing u_m by $\frac{1}{\|u_m\|_{L^2}} u_m$ if necessary, we can assume that $\|u_m\|_{L^2} = 1$.
- Then $\|u_m\|_{L^2} = 1$, $\|f_m\|_{L^q} < \frac{1}{m}$ and by (*), $\|u_m\|_{H^1} \leq C$.
  By the reflexivity of H^1 and Rellich-Kondrachov's theorem, we may assume that $u_m \rightharpoonup u$ in H^1 and $u_m \to u$ in L^2.
  Note that $\|u\|_{L^2} = 1$.
- To conclude, we show that $Lu = 0$, which implies $u = 0$ by hypothesis, and amounts to a contradiction with $\|u\|_{L^2} = 1$.

Energy estimate with L^q right hand side

Proof

- We start with $Lu_m = f_m$ which means

$$\int_{B_1} \Big[a_{ij}\partial_j u_m \partial_i v + b_i \partial_i u_m v + c u_m v\Big]\,dx = \int_{B_1} f_m v\,dx \text{ for all } v \in H_0^1(B_1).$$

  We then send $m \to \infty$ using that $\nabla u_m \rightharpoonup \nabla u$ in L^2, $u_m \to u$ in L^2 and $f_m \to 0$ in L^q to obtain

$$\int_{B_1} \Big[a_{ij}\partial_j u \partial_i v + b_i \partial_i u v + c u v\Big]\,dx = 0 \text{ for all } v \in H_0^1(B_1),$$

  i.e. $Lu = 0$, as desired.

- As $u_m \in H_0^1(B_1)$, we have $u \in H_0^1(B_1)$ and so $u = 0$ by hypothesis. This contradicts the identity $\|u\|_{L^2} = 1$, and finishes the proof.

Non-uniformly elliptic: A case study

Let us now consider an example in $1d$:

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

As $k \to 0$, the ellipticity deteriorates. As $k \to \infty$, the boundedness of k deteriorates.

We have proved 2 estimates:

$$\|u\|_{L^\infty(-1,1)} \leq C_1(k)\|f\|_{L^\infty(-1,1)}, \tag{1}$$

$$\|u\|_{L^\infty(-1,1)} \leq C_2(k)(\|f\|_{L^\infty(-1,1)} + \|u\|_{L^2(-1,1)}). \tag{2}$$

We would now like to have a rough appreciation whether (or how) these constants depend on k, as $k \to 0$ or ∞.

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- We empirically take $f = 1$, so that $\|f\|_{L^\infty} = 1$.
- We know that the problem has uniqueness (why?), so it suffices to find a solution.
- The equation gives $-u'' = 1$ in $(-1,0)$ and $-u'' = 1/k$ in $(0,1)$. So u takes the form

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \alpha(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 + \beta(x-1) & \text{for } x \in (0,1). \end{cases}$$

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- As $u \in H^1(-1,1)$, u is continuous. So

$$-\frac{1}{2} + \alpha = -\frac{1}{2k} - \beta.$$

- As au' is weakly differentiable, it is continuous and so

$$-1 + \alpha = 1 + k\beta.$$

- So we find $\alpha = \frac{k+3}{2(k+1)}$ and $\beta = -\frac{3k+1}{2k(k+1)}$.

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- So we have

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \frac{k+3}{2(k+1)}(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 - \frac{3k+1}{2k(k+1)}(x-1) & \text{for } x \in (0,1). \end{cases}$$

- We find $\|u\|_{L^\infty} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^\infty} \sim 1$ as $k \to \infty$. Therefore

$$C_1(k) \sim \frac{1}{k} \text{ as } k \to 0, \text{ and } C_1(k) \sim 1 \text{ as } k \to \infty.$$

- Similarly $\|u\|_{L^2} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^2} \sim 1$ as $k \to \infty$. Therefore

$$C_2(k) \sim 1 \text{ as } k \to 0, \infty.$$

More examples...

Some other motivating examples you may want to consider:
$a = \chi_{(-1,1)\setminus A} + k\chi_A$ where

- A is an interval of length ε.
- A consists of two or more disjoint intervals of distance ε apart.

Studies of this kind in higher dimensions are active area of research, due to their practical importance.

Local a priori L^∞ estimate

Theorem (Local a priori L^∞ estimates)

Suppose that $a \in L^\infty(B_2)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H^1(B_2) \cap L^\infty(B_2)$ satisfies $Lu = f$ in B_2 in the weak sense for some $f \in L^q(B_2)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_2)} + \|u\|_{L^2(B_2)}),$$

where the constant C depends only on n, q, a.

Local a priori L^∞ estimates

Proof – for awareness and screening of new issues

- We will also use Moser iteration method. Fix some $k > 0$, $p \geq 1$.
- Let $w = u_+ + k$. Unlike in the last lecture, $w^p - k^p$ is no longer in $H_0^1(B_2)$ and so cannot be used directly as a test function.
- To fix the issue, we take a function $\zeta \in C_c^\infty(B_2)$ with $|\zeta| \leq 1$ and use $v = \zeta^2(w^p - k^p)$ as a test function.

  We have

$$\begin{aligned}\int_{B_2} f\, v\, dx &= \int_{B_2} a_{ij}\partial_j u \partial_i v\, dx \\ &= \int_{B_2} p\zeta^2 w^{p-1} a_{ij}\partial_j u \partial_i u_+\, dx \\ &\qquad + \int_{B_2} 2\zeta a_{ij}\partial_j u \partial_i \zeta (w^p - k^p)\, dx,\end{aligned}$$

  where in the rest of the proof red terms indicate new terms that appear due to the introduction of ζ in the proof.

Local a priori L^∞ estimates

Proof

- $$\int_{B_2} f\, v\, dx = \int_{B_2} p\zeta^2 w^{p-1} a_{ij}\partial_j u \partial_i u_+ \, dx + \int_{B_2} 2\zeta a_{ij}\partial_j u \partial_i \zeta (w^p - k^p)\, dx.$$
- The first term on the right hand side is treated using ellipticity as usual:

$$\int_{B_2} p\zeta^2 w^{p-1} a_{ij}\partial_j u \partial_i u_+ \, dx \geq \lambda p \int_{B_2} \zeta^2 w^{p-1} |\nabla u_+|^2 \, dx$$

The left hand side is also treated as last time:

$$\int_{B_2} f\, v\, dx \leq \int_{B_2} \zeta^2 |f| w^p \, dx \leq \int_{B_2} \frac{|f|}{k} \zeta^2 w^{p+1}\, dx.$$

Local a priori L^∞ estimates

Proof

- Putting the inequalities together and rearranging, we thus have

$$p\int_{B_2} \zeta^2 w^{p-1}|\nabla u_+|^2\,dx \leq C\int_{B_2}\frac{|f|}{k}\zeta^2 w^{p+1}\,dx + C\int_{B_2}|\zeta||\nabla u||\nabla\zeta||w^p - k^p|\,dx.$$

  As $w \geq k$, we have $|w^p - k^p| = w^p - k^p < w^p$. Also, in $\{w^p - k^p > 0\} = \{u > 0\}$, we have $\nabla u = \nabla u_+$.
  Therefore

$$p\int_{B_2} \zeta^2 w^{p-1}|\nabla u_+|^2\,dx \leq C\int_{B_2}\frac{|f|}{k}\zeta^2 w^{p+1}\,dx + C\int_{B_2}|\zeta||w|^{\frac{p-1}{2}}|\nabla u_+||\nabla\zeta|w^{\frac{p+1}{2}}\,dx.$$

Local a priori L^∞ estimates

Proof

- $$p \int_{B_2} \zeta^2 w^{p-1} |\nabla u_+|^2 \, dx \le C \int_{B_2} \frac{|f|}{k} \zeta^2 w^{p+1} \, dx + C \int_{B_2} |\zeta| |w|^{\frac{p-1}{2}} |\nabla u_+| w^{\frac{p+1}{2}} |\nabla \zeta| \, dx.$$
- By Cauchy-Schwarz' inequality, we have

the last integral $\le \frac{1}{2} p \int_{B_2} \zeta^2 w^{p-1} |\nabla u_+|^2 \, dx + \frac{C}{p} \int_{B_2} w^{p+1} |\nabla \zeta|^2 \, dx.$

- It follows that

$$p \int_{B_2} \zeta^2 w^{p-1} |\nabla u_+|^2 \, dx \le C \int_{B_2} \Big[\frac{|f|}{k} \zeta^2 + \frac{1}{p} |\nabla \zeta|^2\Big] w^{p+1} \, dx.$$

Local a priori L^∞ estimate

Proof

- Rearranging, we obtain

$$\int_{B_2} \zeta^2 |\nabla w^{\frac{p+1}{2}}|^2 \, dx \leq Cp \int_{B_2} \left[\frac{|f|}{k} \zeta^2 + |\nabla \zeta|^2 \right] w^{p+1} \, dx.$$

- The above inequality gives

$$\|\zeta(w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}})\|_{H^1}^2 \leq Cp \int_{B_2} \left[\frac{|f|}{k} \zeta^2 + \zeta^2 + |\nabla \zeta|^2 \right] w^{p+1} \, dx.$$

- By Gagliardo-Nirenberg-Sobolev's inequality, this implies that

$$\|\zeta(w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}})\|_{L^{\frac{2n}{n-2}}}^2 \leq Cp \int_{B_2} \left[\frac{|f|}{k} \zeta^2 + \zeta^2 + |\nabla \zeta|^2 \right] w^{p+1} \, dx.$$

Local a priori L^∞ estimate

Proof

- $\|\zeta(w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}})\|^2_{L^{\frac{2n}{n-2}}} \leq Cp \int_{B_2} \left[\frac{|f|}{k}\zeta^2 + \zeta^2 + |\nabla\zeta|^2\right] w^{p+1}\,dx.$
- Thus, by triangle inequality,

$$\|\zeta w^{\frac{p+1}{2}}\|^2_{L^{\frac{2n}{n-2}}} \leq Cp \int_{B_2} \left[\frac{|f|}{k}\zeta^2 + \chi_{Supp(\zeta)} + |\nabla\zeta|^2\right] w^{p+1}\,dx.$$

- Using Hölder's inequality, we then arrive at

$$\|\zeta^2 w^{p+1}\|_{L^{\frac{n}{n-2}}(Supp(\zeta))} \leq Cp\Big[\|\frac{|f|}{k}\|_{L^q} + 1 + \|\nabla\zeta\|^2_{L^\infty}\Big]\|w^{p+1}\|_{L^{q'}(Supp(\zeta))}.$$

- We again choose k to be any number larger than $\|f\|_{L^q}$ to obtain

$$\|\zeta^2 w^{p+1}\|_{L^{\frac{n}{n-2}}(Supp(\zeta))} \leq Cp\Big[1 + \|\nabla\zeta\|^2_{L^\infty}\Big]\|w^{p+1}\|_{L^{q'}(Supp(\zeta))}.$$

Local a priori L^∞ estimate

Proof

- $\|\zeta^2 w^{p+1}\|_{L^{\frac{n}{n-2}}(Supp(\zeta))} \leq Cp\left[1 + \|\nabla\zeta\|^2_{L^\infty}\right]\|w^{p+1}\|_{L^{q'}(Supp(\zeta))}$. Recalling that $q > n/2$, we have $q' < \frac{n}{n-2}$. The above inequality is therefore self-improving, though not as strong as last time: If w has a bound in $L^{q'(p+1)}(Supp(\zeta))$, then it has a bound in $L^{\frac{n(p+1)}{n-2}}(\{\zeta \geq 1\})$.
- In particular, if we select $0 < r_2 < r_1 < 2$ and $\zeta \in C_c^\infty(B_{r_1})$ with $\zeta \equiv 1$ in B_{r_2} and $|\nabla\zeta| \leq \frac{C}{r_1 - r_2}$, we have

$$\|w^{p+1}\|_{L^{\frac{n}{n-2}}(B_{r_2})} \leq \frac{Cp}{(r_1 - r_2)^2}\|w^{p+1}\|_{L^{q'}(B_{r_1})}$$

where the constant C is independent of r_1, r_2 and p.

Local a priori L^∞ estimate

Proof

- $\|w^{p+1}\|_{L^{\frac{n}{n-2}}(B_{r_2})} \leq \dfrac{Cp}{(r_1 - r_2)^2}\|w^{p+1}\|_{L^{q'}(B_{r_1})}$.
- As in the last lecture, let $\chi = \frac{n}{(n-2)q'} > 1$ and $t_m = \gamma\chi^m$ for some $\gamma > 2q'$.

If the red terms weren't there then the above would give

$$\begin{aligned}\|w\|_{L^{t_{m+1}}} &\leq (Ct_m)^{\frac{q'}{t_m}}\|w\|_{L^{t_m}}\\ &= (C\gamma)^{q'\gamma^{-1}\chi^{-m}}\chi^{q'\gamma^{-1}m\chi^{-m}}\|w\|_{L^{t_m}}.\end{aligned}$$

Hence by induction,

$$\|w\|_{L^{t_{m+1}}} \leq (C\gamma)^{q'\gamma^{-1}\sum_m \chi^{-m}}\chi^{q'\gamma^{-1}\sum_m m\chi^{-m}}\|w\|_{L^\gamma} \leq C\|w\|_{L^\gamma}.$$

Sending $m \to \infty$ would yield the conclusion.

Local a priori L^∞ estimate

Proof

- $\|w^{p+1}\|_{L^{\frac{n}{n-2}}(B_{r_2})} \leq \dfrac{Cp}{(r_1 - r_2)^2}\|w^{p+1}\|_{L^{q'}(B_{r_1})}$.
- As in the last lecture, let $\chi = \frac{n}{(n-2)q'} > 1$ and $t_m = \gamma\chi^m$ for some $\gamma > 2q'$.
  To accommodate the red terms, we look as radii $r_m = 1 + 2^{-m-1}$. Then

$$\|w\|_{L^{t_{m+1}}(B_{r_{m+1}})} \leq \left(\frac{Ct_m}{2^{-2m}}\right)^{\frac{q'}{t_m}}\|w\|_{L^{t_m}(B_{r_m})}$$
$$= (C\gamma)^{q'\gamma^{-1}\chi^{-m}}(4\chi)^{q'\gamma^{-1}m\chi^{-m}}\|w\|_{L^{t_m}(B_{r_m})}.$$

By induction, we hence get

$$\|w\|_{L^{t_{m+1}}(B_{r_{m+1}})} \leq (C\gamma)^{q'\gamma^{-1}\sum_m \chi^{-m}}(4\chi)^{q'\gamma^{-1}\sum_m m\chi^{-m}}\|w\|_{L^\gamma(B_{3/2})}$$
$$\leq C\|w\|_{L^\gamma(B_{3/2})}.$$

Local a priori L^∞ estimate

Proof

- Sending $m \to \infty$, we obtain

$$\|w\|_{L^\infty(B_1)} \leq C\|w\|_{L^\gamma(B_{3/2})} \text{ when } \gamma > 2q'.$$

- The reduction from L^γ to L^2 in this local case is not as straightforward as before. Let us assume for the moment that it is done so that

$$\|w\|_{L^\infty(B_1)} \leq C\|w\|_{L^2(B_2)}.$$

- We conclude by recalling that $w = u_+ + k$ and k can be any positive constant larger than $\|f\|_{L^q(B_2)}$:

$$\|u_+\|_{L^\infty(B_1)} \leq C\big(\|u\|_{L^2(B_2)} + \|f\|_{L^q(B_2)}\big)$$

- The same argument applies to u_-. The conclusion follows.

Local a priori L^∞ estimate

Proof

- $\|w\|_{L^\infty(B_1)} \leq C\|w\|_{L^\gamma(B_{3/2})}$ when $\gamma > 2q'$.
- We now return to the reduction from L^γ to L^2.
- It turns out that the proof of the first bullet point above yields some constant C and exponent m such that

$$\|w\|_{L^\infty(B_{r_2})} \leq \frac{C}{(r_1 - r_2)^m}\|w\|_{L^\gamma(B_{r_1})} \text{ for all } 0 < r_2 < r_1 < 2.$$

Now we write as last time

$$\|w\|_{L^\gamma(B_{r_1})} \leq \|w\|_{L^\infty(B_{r_1})}^{1-\frac{2}{\gamma}}\|w\|_{L^2(B_{r_1})}^{\frac{2}{\gamma}}$$

so that

$$\|w\|_{L^\infty(B_{r_2})} \leq \frac{C}{(r_1 - r_2)^m}\|w\|_{L^\infty(B_{r_1})}^{1-\frac{2}{\gamma}}\|w\|_{L^2(B_{r_1})}^{\frac{2}{\gamma}} \text{ for all } 0 < r_2 < r_1 < 2.$$

Local a priori L^∞ estimate

Proof

- $\|w\|_{L^\infty(B_{r_2})} \leq \frac{C}{(r_1 - r_2)^m} \|w\|_{L^\infty(B_{r_1})}^{1-\frac{2}{\gamma}} \|w\|_{L^2(B_{r_1})}^{\frac{2}{\gamma}}$ for all $0 < r_2 < r_1 < 2$.
- To proceed, we use the inequality $ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$ on the right hand side to get

$$\begin{aligned}
\|w\|_{L^\infty(B_{r_2})} &\leq \frac{1}{2}\|w\|_{L^\infty(B_{r_1})} + \frac{C}{(r_1 - r_2)^{\hat{m}}}\|w\|_{L^2(B_{r_1})}\\
&\leq \frac{1}{2}\|w\|_{L^\infty(B_{r_1})} + \frac{C}{(r_1 - r_2)^{\hat{m}}}\|w\|_{L^2(B_2)}
\end{aligned}$$

  for all $0 < r_2 < r_1 < 2$.

Local a priori L^∞ estimate

Proof

- We thus have

$$\|w\|_{L^\infty(B_{r_2})} \leq \frac{1}{2}\|w\|_{L^\infty(B_{r_1})} + \frac{C\|w\|_{L^2(B_2)}}{(r_1 - r_2)^{\hat{m}}} \text{ for all } 0 < r_2 < r_1 < 2.$$

- The conclusion follows from the following lemma:

Lemma (Giaquinta-Giusti)

Suppose $Z : [r, R] \to [0, \infty)$ *is a bounded and*

$$Z(s) \leq \frac{1}{2} Z(t) + A(t - s)^{-\alpha} \text{ for all } r \leq s < t \leq R$$

for some constant $A > 0$, $\alpha \geq 0$. *Then, for some* $c = c(\alpha) > 0$,

$$Z(r) \leq c(\alpha) A (R - r)^{-\alpha}.$$

Giaquinta-Giusti's lemma

Proof

- Fix some $\lambda \in (0,1)$ for the moment and let $t_m = R - \lambda^m(R-r)$.
- Then

$$Z(t_m) \leq \frac{1}{2} Z(t_{m+1}) + A[(1-\lambda)\lambda^m(R-r)]^{-\alpha}.$$

- So

$$\begin{aligned} Z(r) = Z(t_0) &\leq \frac{1}{2} Z(t_1) + A[(1-\lambda)(R-r)]^{-\alpha} \\ &\leq \frac{1}{2^2} Z(t_2) + \frac{1}{2} A[(1-\lambda)\lambda^1(R-r)]^{-\alpha} + A[(1-\lambda)(R-r)]^{-\alpha} \\ &\leq \ldots \\ &\leq \frac{1}{2^m} Z(t_m) + A[(1-\lambda)(R-r)]^{-\alpha} \sum_{k=0}^{m-1} 2^{-k} \lambda^{-k\alpha}. \end{aligned}$$

Giaquinta-Giusti's lemma

Proof

- $Z(r) \leq \frac{1}{2^m} Z(t_m) + A[(1-\lambda)(R-r)]^{-\alpha} \sum_{k=0}^{m-1} 2^{-k} \lambda^{-k\alpha}.$
- Sending $m \to \infty$ using that Z is bounded, we hence have

$$Z(r) \leq A[(1-\lambda)(R-r)]^{-\alpha} \sum_{k=0}^{\infty} 2^{-k} \lambda^{-k\alpha}.$$

- Choosing $\lambda \in (0,1)$ such that $2\lambda^\alpha > 1$, we see that the geometric sum converges, giving the lemma.

Other topics

- Homogenization, multi-scale issues (see the case study we did earlier).
- Linear elliptic systems (last year lectures).
- Linear elliptic equations in non-divergence form: A glimpse.

Elliptic systems

Consider a second order linear system of partial differential equation for a function $u = (u_1, \ldots, u_m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$(Lu)_\alpha = -\partial_i(a_{\alpha\beta,ij}\partial_j u_\beta) + \text{ lower order terms} = f_\alpha$$

where repeated Roman indices are summed from 1 to n and repeated Greek indices are summed from 1 to m.

- Ellipticity (Legendre-Hadamard condition): Consideration in the calculus of variation suggests that ellipticity should mean

$$a_{\alpha\beta,ij}\xi_i\xi_j\eta_\alpha\eta_\beta > 0 \text{ for } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m, \xi, \eta \neq 0.$$

- In most case, one requires the stronger condition (strong ellipticity):

$$a_{\alpha\beta,ij}p_{\alpha i}p_{\beta j} > 0 \text{ for } p \in \mathbb{R}^{n\times m}, p \neq 0.$$

- Symmetricity:

$$a_{\alpha\beta,ij} = a_{\beta\alpha,ji}.$$

Elliptic systems

$$(Lu)_\alpha = -\partial_i(a_{\alpha\beta,ij}\partial_j u_\beta) + \text{ lower order terms} = f_\alpha.$$

- Much is understood, but theory is far less complete!
- Weak solutions are defined similarly using vector-valued test functions.
- Under the right condition on the lower order coefficients e.g. absence of first order term and coercivity, existence can be proved for symmetric system by the Riesz representation theorem (under strong ellipticity) or the direct method of the calculus of variations (under Legendre-Hadamard).
- In the absence of lower order terms: The Legendre-Hadamard condition does not imply uniqueness (Edenstein-Fosdick). Strong ellipticity does imply uniqueness.
  In particular, the Fredholm alternative does not hold, namely there exists operator which gives solvability but has no uniqueness.

Elliptic systems

$$(Lu)_\alpha = -\partial_i(a_{\alpha\beta,ij}\partial_j u_\beta) + \text{ lower order terms} = f_\alpha.$$

- H^2 regularity holds under strong ellipticity.
- Hölder continuity needs not hold for solutions to a bounded measurable and strongly elliptic system.

Theorem (Giusti-Miranda)

Let B be the unit ball in $\mathbb{R}^n$, $n \geq 3$ and $u(x) = \frac{x}{|x|}$. Then $u \in H^1(B) \setminus C(B)$ and u satisfies $(Lu)_\alpha = -\partial_i(A_{\alpha\beta,ij}\partial_j u_\beta) = 0$ in B where

$$A_{\alpha\beta,ij} = \delta_{\alpha\beta}\delta_{ij} + \left[\delta_{\alpha i} + \frac{2}{n-2}\frac{x_\alpha x_i}{|x|^2}\right]\left[\delta_{\beta j} + \frac{2}{n-2}\frac{x_j x_\beta}{|x|^2}\right].$$

Elliptic systems

Proof

- By brute force, one check that, for $x \neq 0$, u is smooth and $Lu(x) = 0$.
- Note that at this point one cannot conclude that $Lu = 0$ in the weak sense yet. [One should keep in mind the example that $-\Delta \frac{1}{|x|^{n-2}} = 0$ in $\mathbb{R}^n \setminus 0$ (for $n \geq 3$) but $-\Delta \frac{1}{|x|^{n-2}} \neq 0$ in $\mathbb{R}^n$ in the weak sense.]
- We proceed to show that $Lu = 0$ in B, i.e.

$$\int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha \, dx = 0 \text{ for all } \varphi \in C_c^\infty(B; \mathbb{R}^n).$$

- The fact that $Lu = 0$ in $B \setminus \{0\}$ gives that

$$\int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha \, dx = 0 \text{ for all } \varphi \in C_c^\infty(B \setminus \{0\}; \mathbb{R}^n).$$

Elliptic systems

Proof

- Fix now a function $\varphi \in C_c^\infty(B)$.
  For small $\varepsilon > 0$, take a bump function $\zeta_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that $\zeta_\varepsilon \equiv 0$ in $B_\varepsilon(0)$, $\zeta_\varepsilon \equiv 1$ outside of $B_{2\varepsilon}(0)$, $|\zeta_\varepsilon| \leq 1$ and $|\nabla \zeta_\varepsilon| \leq \frac{C}{\varepsilon}$.
  Let $\varphi^{(\varepsilon)} = \varphi \zeta_\varepsilon \in C_c^\infty(B \setminus \{0\})$.
- As $Lu = 0$ in $B \setminus \{0\}$, we have

$$
\begin{aligned}
0 &= \int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha^{(\varepsilon)}\, dx \\
&= \int_B A_{\alpha\beta,ij} \partial_j u_\beta [\partial_i \varphi_\alpha \zeta_\varepsilon + \varphi_\alpha \partial_i \zeta_\varepsilon]\, dx \\
&= \int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha \zeta_\varepsilon\, dx + \int_B A_{\alpha\beta,ij} \partial_j u_\beta \varphi_\alpha \partial_i \zeta_\varepsilon\, dx \\
&=: I_1 + I_2.
\end{aligned}
$$

Elliptic systems

Proof

- Consider $I_1 = \int_B A_{\alpha\beta,ij}\partial_j u_\beta \partial_i \varphi_\alpha \zeta_\varepsilon \, dx$.
  The integrand is bounded by $|A_{\alpha\beta,ij}\partial_j u_\beta \partial_i \varphi_\alpha|$, which is integrable, and converges a.e. to $A_{\alpha\beta,ij}\partial_j u_\beta \partial_i \varphi_\alpha$ as $\varepsilon \to 0$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0} I_1 = \int_B A_{\alpha\beta,ij}\partial_j u_\beta \partial_i \varphi_\alpha \, dx.$$

- Consider next $I_2 = \int_B A_{\alpha\beta,ij}\partial_j u_\beta \varphi_\alpha \partial_i \zeta_\varepsilon \, dx$.
  Note that $|\nabla \zeta_\varepsilon| \leq \frac{C}{\varepsilon}$ and is supported in $B_{2\varepsilon} \setminus B_\varepsilon$. Furthermore, we have $|\nabla u| = \frac{\sqrt{n-1}}{|x|}$. Hence

$$I_2 \leq \frac{C}{\varepsilon^2}|B_{2\varepsilon} \setminus B_\varepsilon| \leq C\varepsilon^{n-2} \xrightarrow{\varepsilon \to 0} 0.$$

Elliptic systems

Proof

- So we have shown that $0 = I_1 + I_2$,

$$\lim_{\varepsilon \to 0} I_1 = \int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha \, dx.$$

and

$$\lim_{\varepsilon \to 0} I_2 = 0.$$

- We conclude that

$$\int_B A_{\alpha\beta,ij} \partial_j u_\beta \partial_i \varphi_\alpha \, dx = 0.$$

Since φ was selected in $C_c^\infty(B)$ arbitrarily, this means $Lu = 0$ in B in the weak sense.

Non-divergence operators

$$Lu = -a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu = f$$

- Strong solution: One assumes $u \in W^{2,p}$ for some $p \geq 1$. The equation is understood in the almost everywhere sense. If $p > n$, then u is twice classically differentiable almost everywhere, so those appears rather natural.
- Existence: Assume a is continuous, $b, c \in L^\infty$, $c \leq 0$. Then for every $f \in L^p$ and $u_0 \in W^{2,p}$, there exists a unique $u \in W^{2,p}$ such that $Lu = f$ and $u - u_0 \in W_0^{1,p}$.
- Estimate and regularity: If $Lu \in L^q$ and $u_0 \in W^{2,q}$ with $q \geq p$, then $u \in W^{2,q}$ with

$$\|u\|_{W^{2,q}} \leq C(\|f\|_{L^q} + \|u_0\|_{W^{2,q}}).$$

Non-divergence operators

$$Lu = -a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu = f$$

- Viscosity solution: One tests the equation from above and below using approximate paraboloid. Suitable for fully nonlinear. Doesn't requires much regularity.
- Krylov-Safonov's theorem: If $a, b, c \in L^\infty$, uniformly elliptic, $f \in L^n$, then u is Hölder continuous. (So this is the equivalence of De Giorgi-Moser-Nash' theorem but with a stronger assumption on f.) Proof much trickier.
- Alexandrov-Bakelman-Pucci estimate: If $Lu \geq f$, $u \in C^0 \cap W^{2,n}$, then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\|f\|_{L^n}.$$

Sometimes can lower L^n to $L^{n-\varepsilon}$ with ε depending on uniform ellipticity. Cannot be universally lowered.