B3.3 Algebraic Curves

Hilary 2022

kirwan@maths.ox.ac.uk

Example sheet 1

Section A (introductory questions, not for marking, solutions available)

1. Let $(2,1),(1,1),(3,4)$ represent points in the projective line $\mathbb{P}^1(\mathbb{R})$. Find representative vectors v_1, v_2, v_3 for these points which satisfy $v_1+v_2+v_3=0$.

Solution. We want $a,b,c \in \mathbb{R}\setminus\{0\}$ such that $a(2,1)+b(1,1)+c(3,4)=0$. We can take $a=c=1$ and $b=-5$, giving $v_1=(2,1)$, $v_2=(-5,-5)$ and $v_3=(3,4)$.

2. Embed $\mathbb{R}^2$ in the projective plane $\mathbb{RP}^2$ by $(x,y)\mapsto[1,x,y]$. Find the point of intersection in $\mathbb{RP}^2$ of the projective lines corresponding to the parallel lines $y=mx$ and $y=mx+c$ in $\mathbb{R}^2$.

Solution. In homogeneous coordinates the projective lines have equations $mx_1-x_2=0$ and $cx_0+mx_1-x_2=0$, which meet at $[0,1,m]$, a 'point at infinity' in the projective plane.

Section B (questions to be handed in for marking)

3. Embed $\mathbb{C}^2$ in the projective plane $\mathbb{CP}^2$ by $(x,y)\mapsto[1,x,y]$. Find the points of intersection in $\mathbb{CP}^2$ of the projective line and projective curve corresponding to the line $y=mx+c$ and curve $y^2=ax^2+b$ in $\mathbb{C}^2$, identifying any special cases.

4. Let $\mathbb{Z}_2$ be the field $\{0,1\}$ of integers modulo 2. Show that the number of points in n-dimensional projective space over $\mathbb{Z}_2$ is $2^{n+1}-1$. How many projective lines are there in this space?

5. What are the answers to Q4 if you instead work over the field $\mathbb{Z}_p$ with p elements, where p is an odd prime?

6. Show that $f:([z_0,z_1],[w_0,w_1])\mapsto[z_0w_0,-z_0w_1-z_1w_0,z_1w_1]$ is a well-defined map from $\mathbb{CP}^1\times\mathbb{CP}^1$ to $\mathbb{CP}^2$.

Also show that this map is surjective.

Is the corresponding map $f:\mathbb{RP}^1\times\mathbb{RP}^1\to\mathbb{RP}^2$ surjective?

7. Show that complex projective space $\mathbb{CP}^n$ is compact. What is the relationship between this space and a sphere of appropriate dimension?

Section C (optional extension questions, not to be handed in for marking)

8. Let $T:V\to V$ be an invertible linear map. When is $[v]\in\mathbb{P}(V)$ a fixed point of the projective transformation $\tau:\mathbb{P}(V)\to\mathbb{P}(V)$ defined by T? Show that every projective transformation τ of $\mathbb{RP}^2$ has a fixed point.