B3.3 Algebraic Curves

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Example sheet 2

Section A (introductory questions, not for marking, solutions available)

1. Let C be the projective curve with equation

$$x^2 + y^2 = z^2.$$

Show that the projective line through the points $[0, 1, 1]$ and $[t, 0, 1]$ meets C in the two points $[0, 1, 1]$ and $[2t, t^2 - 1, t^2 + 1]$.

Show that there is a bijection α between the projective line $y = 0$ and C given by:

$$[t, 0, 1] \mapsto [2t, t^2 - 1, t^2 + 1]$$

$$[1, 0, 0] \mapsto [0, 1, 1].$$

Solution. A point on the projective line through $[0, 1, 1]$ and $[t, 0, 1]$ is given by $[at, b, a+b]$ where a and b are not both 0. This point lies on C iff $a^2t^2 + b^2 = a^2 + 2ab + b^2$, and hence iff $a^2(t^2 - 1) = 2ab$. This happens iff either $a = 0$, in which case the point is $[0, 1, 1]$, or $a(t^2 - 1) = 2b$, in which case the point is $[2t, t^2 - 1, t^2 + 1]$.

The projective line $y = 0$ consists of the points $[t, 0, 1]$ together with $[1, 0, 0]$. The projective line through the points $[0, 1, 1]$ and $[1, 0, 0]$ is the tangent line to C at $[0, 1, 1]$ and intersects C at this point only (with multiplicity two). The map α has an inverse which sends a point p of $C \setminus \{[0, 1, 1]\}$ to the unique point of intersection of the projective line through p and $[0, 1, 1]$ and the projective line $y = 0$, and sends $[0, 1, 1]$ to the unique point of intersection of the tangent line to C at $[0, 1, 1]$ and the projective line $y = 0$, which is $[1, 0, 0]$. Thus α is bijective.

Section B (questions to be handed in for marking)

2. Show that a homogeneous polynomial in two variables x, y may be factored into linear polynomials over $\mathbb{C}$.

3. Let C be the affine curve in $\mathbb{C}^2$ defined by a polynomial $P(x, y)$. The *multiplicity* of C at a point $(a, b) \in C$ is defined to be the smallest positive integer m such that

$$\frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \neq 0$$

for some $i, j \in \mathbb{N}$ with $i + j = m$. The polynomial

$$\sum_{i+j=m} \frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \frac{(x - a)^i (y - b)^j}{i!j!}$$

is then homogeneous of degree m in $x-a$ and $y-b$, so is a product of m linear polynomials of the form $\alpha(x-a)+\beta(y-b)$ for $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$.The m lines (not necessarily distinct) defined by these linear polynomials are the *tangent lines* to C at (a,b).

Show that $(a,b)\in C$ is a nonsingular point of C if and only if its multiplicity is 1; what is its (unique) tangent line in this case?

A singular point of C of multiplicity m is called an *ordinary singular point* if the homogeneous polynomial of degree m defining its tangent lines has no repeated factors (that is, the point has m distinct tangent lines). A singular point of C of multiplicity 2 (respectively 3) is called a *double point* (respectively a *triple point*).

Find the multiplicities of the following curves at the origin, and determine whether the origin is an ordinary singular point in each case:

(i) $y^2=x^3+x^2$;

(ii) $y^2=x^3$;

(iii) $(x^4+y^4)^2=x^2y^2$;

(iv) $(x^4+y^4-x^2-y^2)^2=9x^2y^2$.

4. Show that if $\alpha_1,\ldots,\alpha_r$ are *distinct*, then the affine curve

$$y^2=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_r)$$

is nonsingular.

What can you say about the associated projective curve?

5. (i) Show that the affine curve $y^2=x^3+x$ in $\mathbb{C}^2$ is nonsingular.

(ii) Now consider this curve over the finite field $\mathbb{Z}_p$ where p is a prime. That is, we consider the curve in $(\mathbb{Z}_p)^2$ with equation $y^2=x^3+x$.

For which p is this nonsingular?

Section C (optional extension questions, not to be handed in for marking)

6. Let $\mathbb{P}(V)$ be an n-dimensional projective space. We say that $n+2$ points in $\mathbb{P}(V)$ are in *general position* if each subset of $n+1$ points has linearly independent representative vectors.

Show that if $X_1,\ldots,X_{n+2}$ are in general position then we can choose representative vectors $v_1,\ldots,v_{n+2}$ such that

$$v_{n+2}=\sum_{i=1}^{n+1}v_i.$$

Deduce that given another set $Y_1,\ldots,Y_{n+2}$ of points in general position there exists a unique projective transformation τ such that

$$\tau: X_i\mapsto Y_i \qquad (i=1,\ldots,n+2).$$

What familiar result does this yield if $n=1$?

7. Prove Pappus's theorem by using the notion of general position as follows. First prove the theorem in the degenerate case when A,B,C',B' are *not* in general position. Then assume these points *are* in general position and take them to be $[1,0,0],[0,1,0],[0,0,1],[1,1,1]$. Calculate the three intersections explicitly, verify they are collinear, and explain why this proves the theorem in general.