B3.3 Algebraic Curves

Hilary 2022

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Example sheet 3

Section A (introductory questions, not for marking, solutions available)

1. (i) Show that given any 5 points in $\mathbb{CP}^2$, there is at least one conic passing through them. Show also that this conic is unique if no three of the points are collinear.

(ii) Let C be a quartic (degree 4) curve in $\mathbb{CP}^2$ with four singular points. Use the strong form of Bézout's theorem to show C must be reducible.

(iii) Show that $y^4 - 4xzy^2 - xz(x-z)^2 = 0$ defines a quartic with three singular points.

Solution

(i) A conic is given by an equation of the form $\sum_{j=1}^{3} B_{ij}x_ix_j = 0$ where B is 3 3 symmetric and nonzero. We therefore have 5 homogeneous linear equations in 6 variables so there is a nonzero solution.

Under the non-collinearity assumption, any four of the points are in general position so can be moved to $[1,0,0],[0,1,0],[0,0,1],[1,1,1]$. Let the fifth point be $[a,b,c]$. If B is the symmetric matrix defining the conic we see that

$$B_{11} = B_{22} = B_{33} = 0,$$

$$B_{12} + B_{23} + B_{31} = 0,$$

and

$$abB_{12} + bcB_{23} + caB_{31} = 0.$$

If the coefficients of the last equation are all equal then either a = b = c or two of a, b, c are zero, and in each case this contradicts the assumption that the five points are distinct. So any two nonzero solutions B are scalar multiples of each other, giving a unique conic.

(ii) Let the quartic C have 4 singular points, and let p be a distinct fifth point on C. Let D be a conic through these 5 points, as guaranteed by part (i). Now we apply the strong form of Bézout's theorem:

$$\sum_{p\in C\cap D} I_p(C,D) \geq 1+2+2+2+2 = 9$$

as $I_p(C,D) > 1$ at a singularity of C. But $(\deg C)(\deg D) = 8$, so we have a contradiction unless there is a common component, which implies that C is reducible.

(iii) Let $P(x,y,z) = y^4 - 4xzy^2 - xz(x-z)^2$, so for a singular point we need

$$P_x = -4zy - z(x-z) - 2xz(x-z) = 0,$$

$$P_z = -4xy - x(x-z) + 2xz(x-z) = 0,$$

$$P_y = 4y - 8xyz = 0.$$

The second equation implies either $y = 0$ or $y^2 = 2xz$. If $y = 0$ then (from $P = 0$) we have $x = 0$ or $z = 0$ or $x = z$. The only possible singular point we obtain in this way is $[1, 0, 1]$. If $y^2 = 2xz$ then

$$P_x = -8xz - z(x - 2xz + z) + 2xz - 2xz = -z - 4xz - 3xz$$

which simplifies to $Px = -z(x + z)(3x + z)$. Similarly, we find $P_z = -x(x + 3z)(x + z)$. We now find the resulting singularities are just those with $x + z = 0$ which are

$$[1, i\sqrt{2}, -1] \text{ and } [1, -i\sqrt{2}, -1],$$

giving three singularities in all.

Section B (questions to be handed in for marking)

2. Let $P(x, y, z)$ be a homogeneous polynomial of degree d defining a nonsingular curve C.

(i) Write down Euler's relation for P, P_x, P_y, P_z. Deduce that the Hessian determinant satisfies:

$$z\mathcal{H}_P(x, y, z) = (d - 1) \det \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_x & P_y & P_z \end{pmatrix}.$$

(ii) Deduce further that:

$$z^2\mathcal{H}_P(x, y, z) = (d - 1)^2 \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d - 1) \end{pmatrix}.$$

(iii) Deduce that if $P(x, y, 1) = y - g(x)$ then $[a, b, 1]$ is a flex of C iff $b = g(a)$ and $g''(a) = 0$.

3. Let C and D be nonsingular projective curves of degree n and m in $\mathbb{P}^2$. Show that if C is homeomorphic to D then either $n = m$ or $\{n, m\} = \{1, 2\}$.

4. Show that if C is the conic $y^2 = xz$ then the map

$$f : \mathbb{P}^1 \to C$$

given by

$$f : [s, t] = [s^2, st, t^2]$$

is a homeomorphism.

Deduce without using the degree-genus formula that all nonsingular conics have genus zero.

5. Let $f : X \to Y$ be a (nonconstant) holomorphic map of compact connected Riemann surfaces, where X is the Riemann sphere. Show that Y is homeomorphic to X.

Section C (optional extension questions, not to be handed in for marking)

6(i) Let U be a connected open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic. Show that if $a \in U$, then for sufficiently small real positive r, we have:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

(ii) Deduce that if $|f|$ has a local maximum at $a \in U$, then $|f|$ is constant on some neighbourhood of a.

(iii) Deduce that if $|f|$ has a local maximum at $a \in U$, then f is constant on U.

(iv) Now suppose S is a compact connected Riemann surface and $f : S \mapsto \mathbb{C}$ is a holomorphic function. Show that f is constant. [*You may assume the Identity Theorem for Riemann surfaces; that is, if two holomorphic maps on a (connected) Riemann surface agree on a nonempty open set, then they agree everywhere*].

7. Prove Pascal's Theorem that the pairs of opposite sides of a hexagon inscribed in an irreducible conic meet in three collinear points.