Perturbation Methods : Problem Sheet 1

Q1 (a) $\frac{a_{n+1}(\varepsilon)}{a_n(\varepsilon)} \to 0$ or $a_{n+1}(\varepsilon) = o(a_n(\varepsilon))$ as $\varepsilon \to 0$ $\forall n \in \mathbb{N}_0$

(b) $\frac{f(\varepsilon) - \sum_{n=0}^{N} a_n(\varepsilon)}{a_N(\varepsilon)} \to 0$ or $f(\varepsilon) - \sum_{n=0}^{N} a_n(\varepsilon) = o(a_N(\varepsilon))$ as $\varepsilon \to 0$ $\forall N \in \mathbb{N}_0$

(c) Since $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ for $|x| < 1$,

$$\log(1 - \log\varepsilon) = \log\left(1 + \log\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$= \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \log\left(1 + \frac{1}{\log\left(\frac{1}{\varepsilon}\right)}\right)$$

$$\sim \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n} \quad \text{as } \varepsilon \to 0^+$$

$$\Rightarrow \quad a_0 = \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right), \quad a_n = \frac{(-1)^{n+1}}{n\left(\log\left(\frac{1}{\varepsilon}\right)\right)^n} \quad \text{for } n \in \mathbb{N}$$

(d) Since $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$,

$$\exp\left(-\frac{1}{\varepsilon^2 + \varepsilon^3}\right) = \exp\left(-\frac{1}{\varepsilon^2} \sum_{n=0}^{\infty} (-1)^n \varepsilon^n\right)$$

$$= \exp\left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \varepsilon^n\right)$$

$$= \exp\left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 1\right) \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} \varepsilon^n\right)$$

$$= \exp\left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 1\right) \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} (-1)^{n+1} \varepsilon^n\right)^m \quad \text{for } |\varepsilon| < 1$$

$$\Rightarrow \quad a_n = b_n \varepsilon^n \exp\left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 1\right) \text{ for } n \in \mathbb{N}_0, \text{ where } b_n = O(1) \text{ as } \varepsilon \to 0$$

Q2(a) $x^3 + x - \varepsilon = 0$ as $\varepsilon \to 0$

<u>Iterative method</u>

$\varepsilon = 0 \Rightarrow x = 0, \pm i$

For root near 0 : $x = \varepsilon - x^3$ gives iterative procedure $x_{n+1} = \varepsilon - x_n^3$ $(n \in \mathbb{N}_0)$, $x_0 = 0$.

$$\Rightarrow x_1 = \varepsilon$$

$$x_2 = \varepsilon - \varepsilon^3$$

$$x_3 = \varepsilon - (\varepsilon - \varepsilon^3)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots$$

$$x_4 \sim \varepsilon - (\varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots$$

$$\Rightarrow \underline{x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots \text{ as } \varepsilon \to 0}$$

For root near $x = \pm i$: $x = \pm i\left(1 - \frac{\varepsilon}{x}\right)^{1/2}$ gives iterative procedure procedure $x_{n+1} = \pm i\left(1 - \frac{\varepsilon}{x_n}\right)^{1/2}$ $(n \in \mathbb{N}_0)$, $x_0 = \pm i$.

$$\Rightarrow x_1 = \pm i\left(1 \mp \frac{\varepsilon}{i}\right)^{1/2} \sim \pm i\left(1 \mp \frac{\varepsilon}{2i} + \dots\right) \sim \pm i - \frac{\varepsilon}{2} + \dots$$

$$x_2 \sim \pm i\left(1 \mp \frac{\varepsilon}{i\left(\pm i - \frac{\varepsilon}{2} + \dots\right)}\right) \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots$$

$$x_3 \sim \pm i\left(1 \mp \frac{\varepsilon}{i\left(\pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots\right)}\right) \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots$$

$$\Rightarrow \underline{x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots \text{ as } \varepsilon \to 0}$$

Expansion method

$$x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots \quad \text{as } \varepsilon \to 0$$

$$\Rightarrow \left[x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\right]^3 + \left[x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\right] - \varepsilon = 0$$

$$O(\varepsilon^0): \quad x_0^3 + x_0 = 0 \quad \Rightarrow \quad x_0 = 0, i, -i$$

$$O(\varepsilon^1): \quad 3x_0^2 x_1 + x_1 = 1 \quad \Rightarrow \quad x_1 = \frac{1}{3x_0^2+1} = 1, -\frac{1}{2}, -\frac{1}{2}$$

$$O(\varepsilon^2): \quad 3x_0^2 x_2 + 3x_0 x_1^2 + x_2 = 0 \quad \Rightarrow x_2 = -\frac{3x_0 x_1^2}{3x_0^2+1} = 0, \frac{3i}{8}, \frac{-3i}{8}$$

Hence, for roots near $x = \pm i$, obtain

$$x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots \quad \text{as } \varepsilon \to 0 \quad \text{as before}$$

For root near $x = 0$, need to proceed to higher order to get first three nonzero terms:

$$\left[\varepsilon x_1 + \varepsilon^3 x_3 + \dots\right]^3 + \left[\varepsilon x_1 + \varepsilon^3 x_3 + \dots\right] - \varepsilon = 0$$

$$O(\varepsilon^3): \quad x_1^3 + x_3 = 0 \quad \Rightarrow \quad x_3 = -x_1^3 = -1$$

$$O(\varepsilon^4): \quad x_4 = 0 \quad \Rightarrow \quad x_4 = 0$$

$$O(\varepsilon^5): \quad 3x_1^2 x_3 + x_5 = 0 \quad \Rightarrow \quad x_5 = -3x_1^2 x_3 = 3$$

$$\Rightarrow \quad x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots \quad \text{as } \varepsilon \to 0 \quad \text{as before}$$

(b) $\varepsilon^3 x^2 + \varepsilon x + 1 = 0$ as $\varepsilon \to 0$

Since $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$ for $|x| < 1$,

$$x = \frac{-1 \pm (1-4\varepsilon)^{1/2}}{2\varepsilon^2}$$

$$= \frac{1}{2\varepsilon^2}\left[-1 \pm \left(1 - 2\varepsilon - 2\varepsilon^2 - 4\varepsilon^3 + \dots\right)\right] \quad \text{for } |\varepsilon| < \tfrac{1}{4}$$

$$\Rightarrow x \sim \begin{cases} -\frac{1}{\varepsilon} - 1 - 2\varepsilon + \dots \\ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + 1 + \dots \end{cases} \quad \text{as } \varepsilon \to 0$$

and these expansions converge for $|\varepsilon| < \frac{1}{4}$.

Check via rescaling and expanding: balance 1st and 2nd term $\Rightarrow$ scale $x = \frac{X}{\varepsilon^2} \Rightarrow X^2 + X + \varepsilon = 0$.

$X \sim X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots$ as $\varepsilon \to 0$ $\Rightarrow$

$O(\varepsilon^0)$: $X_0^2 + X_0 = 0 \Rightarrow X_0 = -1, 0$

$O(\varepsilon^1)$: $2X_0X_1 + X_1 = -1 \Rightarrow X_1 = 1, -1$

$O(\varepsilon^2)$: $2X_0X_2 + X_1^2 + X_2 = 0 \Rightarrow X_2 = 1, -1$

$O(\varepsilon^3)$: $2X_0X_3 + 2X_1X_2 + X_3 = 0 \Rightarrow X_3 = 2, -2$

giving same expansions as above.

Note that no other scaling would regularize this problem.

(c) $\varepsilon^2 x^3 + x^2 + 2x + \varepsilon = 0$ as $\varepsilon \to 0$.

Regularize by scaling $x = X/\varepsilon^2$ (balancing 1^{st} and 2^{nd} terms)

$\Rightarrow X^3 + X^2 + 2\varepsilon^2 X + \varepsilon^5 = 0$

$X \sim X_0 + \varepsilon X_1 + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $X_0^3 + X_0^2 = 0 \Rightarrow X_0 = 0, 0, -1$
$O(\varepsilon^1)$: $3X_0^2 X_1 + 2X_0 X_1 = 0 \Rightarrow X_1 = ?, ?, 0$
$O(\varepsilon^2)$: $3X_0^2 X_2 + 3X_0 X_1^2 + 2X_0 X_2 + 2X_0 = 0 \Rightarrow X_2 = ?, ?, 2$

$\Rightarrow$ one root $\underline{x \sim -\frac{1}{\varepsilon^2} + 2 + \dots \text{ as } \varepsilon \to 0}$, but need to rescale for other roots

Balance 2^{nd} and 3^{rd} terms $\Rightarrow \exists$ root $x = O(1) \Rightarrow$

$x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $x_0^2 + 2x_0 = 0 \Rightarrow x_0 = 0, -2$
$O(\varepsilon^1)$: $2x_0 x_1 + 2x_1 = -1 \Rightarrow x_1 = -\frac{1}{2}, \frac{1}{2}$
$O(\varepsilon^2)$: $x_0^3 + 2x_0 x_2 + x_1^2 + 2x_2 = 0 \Rightarrow x_2 = -\frac{1}{8}, -\frac{31}{8}$

$\Rightarrow$ two other roots are

$$\underline{x \sim -2 + \tfrac{1}{2}\varepsilon + \dots, \quad x \sim -\tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \dots \text{ as } \varepsilon \to 0}$$

Note that smallest root balances 3^{rd} and 4^{th} terms: with $x = \varepsilon X$, $\varepsilon^4 X^3 + \varepsilon X^2 + 2X + 1 = 0$, giving $X \sim -\frac{1}{2} + \dots$ as $\varepsilon \to 0$.

Q3(a) (i) $x^3 + \varepsilon(ax+b) = 0$ as $\varepsilon \to 0$ with $a, b = O(1)$

All roots small $x = o(1) \Rightarrow \varepsilon a x = o(\varepsilon)$ as $\varepsilon \to 0$

$\Rightarrow x^3 \sim -b\varepsilon + \dots$ as $\varepsilon \to 0$

$\Rightarrow x \sim e^{\frac{2n\pi i}{3}}(-\varepsilon b)^{1/3} + \dots$ as $\varepsilon \to 0$ for $n = 0, 1, 2$.

(ii) $\varepsilon x^3 + ax + b = 0$ as $\varepsilon \to 0$ with $a, b = O(1)$

Balance 2nd and 3rd term $\Rightarrow$ one root $x \sim \frac{-b}{a} + \dots$ as $\varepsilon \to 0$.

Balance 1st and 2nd term $\Rightarrow$ scale $x = \frac{X}{\varepsilon^{1/2}} \Rightarrow X^3 + aX + b\varepsilon^{1/2} = 0$

$\Rightarrow$ two other roots $X \sim \pm(-a)^{1/2} + \dots$ as $\varepsilon \to 0$

Hence, $x \sim \left(\frac{-b}{a}\right) + \dots, \pm\left(\frac{-a}{\varepsilon}\right)^{1/2} + \dots$ as $\varepsilon \to 0$

(b) $\sqrt{2}\sin\left(\frac{\pi}{4} + x\right) = \sin x + \cos x$
$= x - x^3/3! + 1 - x^2/2! + x^4/4! + O(x^5)$ as $\varepsilon \to 0$

$\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) - 1 - x + x^2/2 = -x^3/6 + x^4/24 + O(x^5)$ as $\varepsilon \to 0$

$\Rightarrow -\frac{x^3}{6} + \frac{x^4}{24} + O(x^6) = -\frac{\varepsilon}{6}$ as $\varepsilon \to 0$

Scale $x = \varepsilon^{1/3} X \Rightarrow X^3 - \frac{\varepsilon^{1/3}}{4} X^4 + O(\varepsilon) = 1$ as $\varepsilon \to 0$

$X \sim X_0 + \varepsilon^{1/3} X_1 + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $X_0^3 = 1 \Rightarrow X_0 = 1$ (real root only)
$O(\varepsilon^{1/3})$: $3X_0^2 X_1 = \frac{1}{4} X_0^4 \Rightarrow X_1 = 1/12$

Hence, $x \sim \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3} + \dots$ as $\varepsilon \to 0$

(c) $a_0(\varepsilon) = \log(1/\varepsilon)$, $a_1(\varepsilon) = \log a_0(\varepsilon)$, ..., $a_{n+1}(\varepsilon) = \log a_n(\varepsilon)$ $(n \in \mathbb{N}_0)$

$$\Rightarrow \quad \frac{a_{n+1}(\varepsilon)}{a_n(\varepsilon)} = \frac{\log a_n(\varepsilon)}{a_n(\varepsilon)} \to 0 \text{ as } a_n(\varepsilon) \to \infty \text{, i.e. as } \varepsilon \to 0^+$$

True $\forall n \in \mathbb{N}_0 \Rightarrow \{a_n(\varepsilon)\}_{n \in \mathbb{N}_0}$ an asymptotic sequence.

$x = \varepsilon \log(1/x)$ as $\varepsilon \to 0^+$.

As $x \to 0$, $\log(1/x)$ varies more slowly than x, so try

$$x_{n+1} = \varepsilon \log(1/x_n), \quad x_0 = \varepsilon \Rightarrow$$

$$x_1 = \varepsilon \log(1/\varepsilon)$$

$$x_2 = \varepsilon \log\left(\frac{1}{\varepsilon \log(\frac{1}{\varepsilon})}\right)$$

$$= \varepsilon \log(\tfrac{1}{\varepsilon}) - \varepsilon \log(\log(\tfrac{1}{\varepsilon}))$$

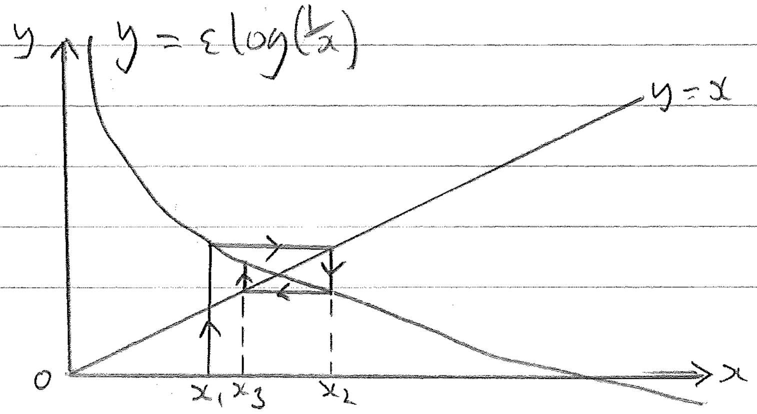
$$x_3 = \varepsilon \log\left(\frac{1}{\varepsilon \log(\frac{1}{\varepsilon})\left(1 - \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})}\right)}\right)$$

$$= \varepsilon \log(\tfrac{1}{\varepsilon}) - \varepsilon \log(\log(\tfrac{1}{\varepsilon})) + \varepsilon \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + O\left(\varepsilon\left(\frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})}\right)^2\right)$$

Need to go to x_4 to make sure first 3 terms stay fixed

$$\Rightarrow \quad x \sim \varepsilon \log(\tfrac{1}{\varepsilon}) - \varepsilon \log(\log(\tfrac{1}{\varepsilon})) + \varepsilon \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + \dots \text{ as } \varepsilon \to 0^+$$

NB:



Q4(a) $$I(\varepsilon) = \frac{e^{1/\varepsilon}}{\varepsilon}\int_{1/\varepsilon}^{\infty} \underbrace{\frac{1}{t}}_{f} \underbrace{e^{-t}}_{g'} dt$$

$$= \frac{e^{1/\varepsilon}}{\varepsilon}\left[\underbrace{\frac{1}{t}}_{f} \underbrace{-e^{-t}}_{g} \Big|_{1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} \underbrace{\frac{-1}{t^2}}_{f'} \underbrace{-e^{-t}}_{g} dt \right]$$

$$= \frac{e^{1/\varepsilon}}{\varepsilon}\left[e^{-1/\varepsilon}\varepsilon - \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^2} dt \right]$$

$\Rightarrow$ true for $N=1$

Assume true for N, then inductive step is

$$(-1)^N N! \int_{1/\varepsilon}^{\infty} \underbrace{\frac{1}{t^{N+1}}}_{f} \underbrace{e^{-t}}_{g'} dt = (-1)^N N! \left[\underbrace{\frac{1}{t^{N+1}}}_{f} \underbrace{-e^{-t}}_{g} \Big|_{1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} \underbrace{\frac{-(N+1)}{t^{N+2}}}_{f'} \underbrace{-e^{-t}}_{g} dt \right]$$

$$= (-1)^N N! \, e^{-1/\varepsilon} \varepsilon^{N+1} + (-1)^{N+1}(N+1)! \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+2}} dt$$

$\Rightarrow$ true for $N+1$, so true $\forall n \in \mathbb{N}$ by induction. $\square$

(b) $$\left| \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+1}} dt \right| < \varepsilon^{N+1} \int_{1/\varepsilon}^{\infty} e^{-t} dt = e^{-1/\varepsilon} \varepsilon^{N+1}$$

$$\Rightarrow \left| I(\varepsilon) - \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \right| = \left| (-1)^N N! \frac{e^{1/\varepsilon}}{\varepsilon} \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N+1}} dt \right|$$

$$< N! \, \varepsilon^N$$

$$\Rightarrow \frac{\left| I(\varepsilon) - \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \right|}{\left| (-1)^{N-1} (N-1)! \, \varepsilon^{N-1} \right|} < N\varepsilon \to 0 \text{ as } \varepsilon \to 0 \ \forall N \in \mathbb{N}$$

$$\Rightarrow I(\varepsilon) \sim \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^n$$

$\square$

(c) $S_N(\varepsilon) = \sum_{n=0}^{N-1} (-1)^n n! \varepsilon^n \to \infty$ as $N \to \infty$ $\forall_{\varepsilon > 0}$.

$|I(0.2) - S_N(0.2)|$ minimal for $N = 5$.

$|I(0.1) - S_N(0.1)|$ minimal for $N = 10$.

See plots over page.

Note that these values correspond to truncating at the smallest term $a_n(\varepsilon) = (-1)^n n! \varepsilon^n$ (called optimal truncation) because the ratio of successive terms $|a_n(\varepsilon) / a_{n-1}(\varepsilon)| = n\varepsilon$ begins to grow when $n > 1/\varepsilon$.

Given $0 < \varepsilon \ll 1$, optimal truncation truncates at $N(\varepsilon)$, where $N(\varepsilon)\varepsilon \leq 1 < (N(\varepsilon)+1)\varepsilon$.

Remainder

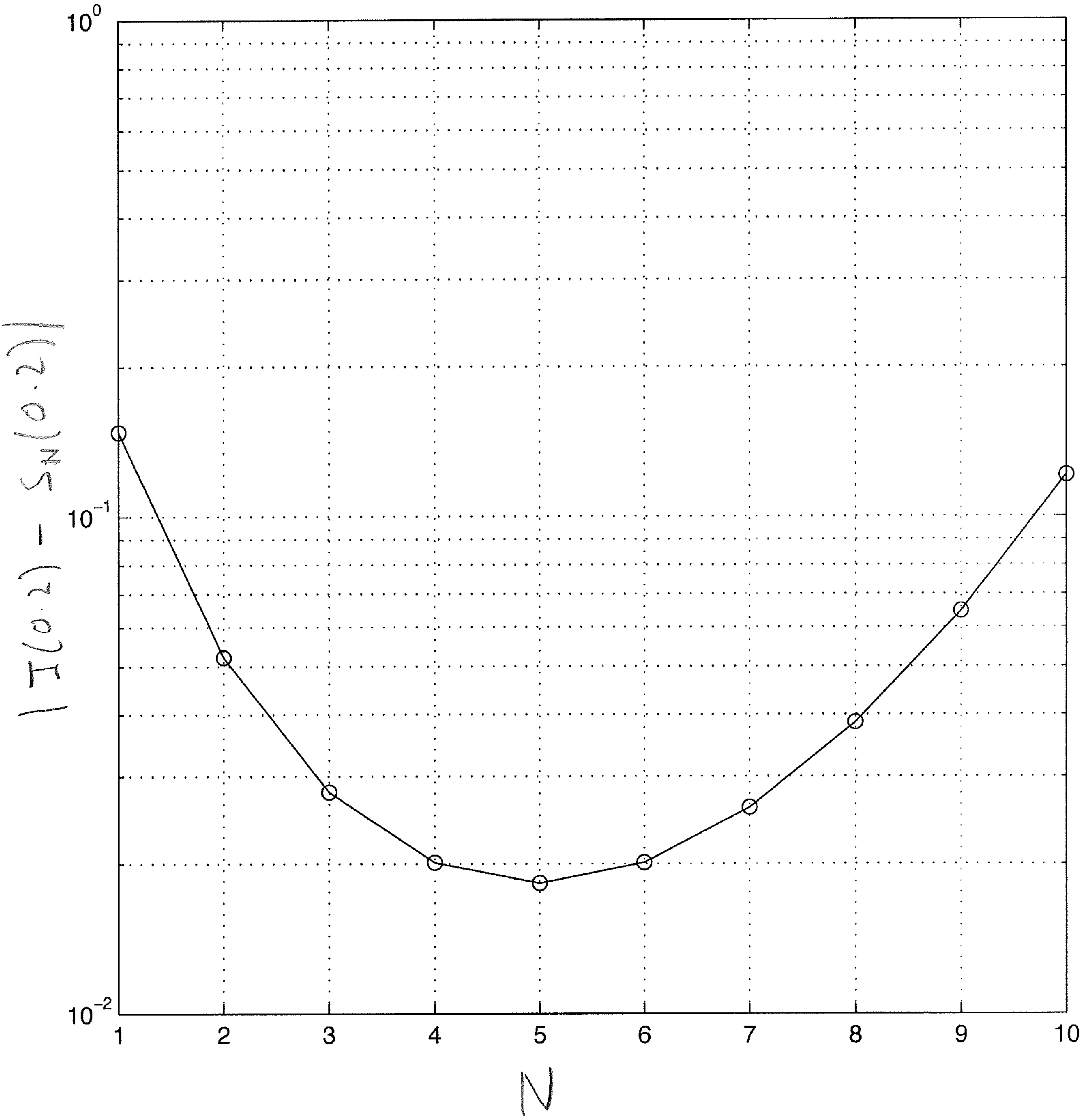
$$R_{N(\varepsilon)}(\varepsilon) = \left| (-1)^{N(\varepsilon)} N(\varepsilon)! \frac{e^{1/\varepsilon}}{\varepsilon} \int_{1/\varepsilon}^{\infty} \frac{e^{-t}}{t^{N(\varepsilon)+1}} dt \right|$$

$$\sim \sqrt{\frac{\pi}{2\varepsilon}}\, e^{-1/\varepsilon} \quad \text{as} \quad \varepsilon \to 0^+$$

via Laplace's method (see sheet 2)

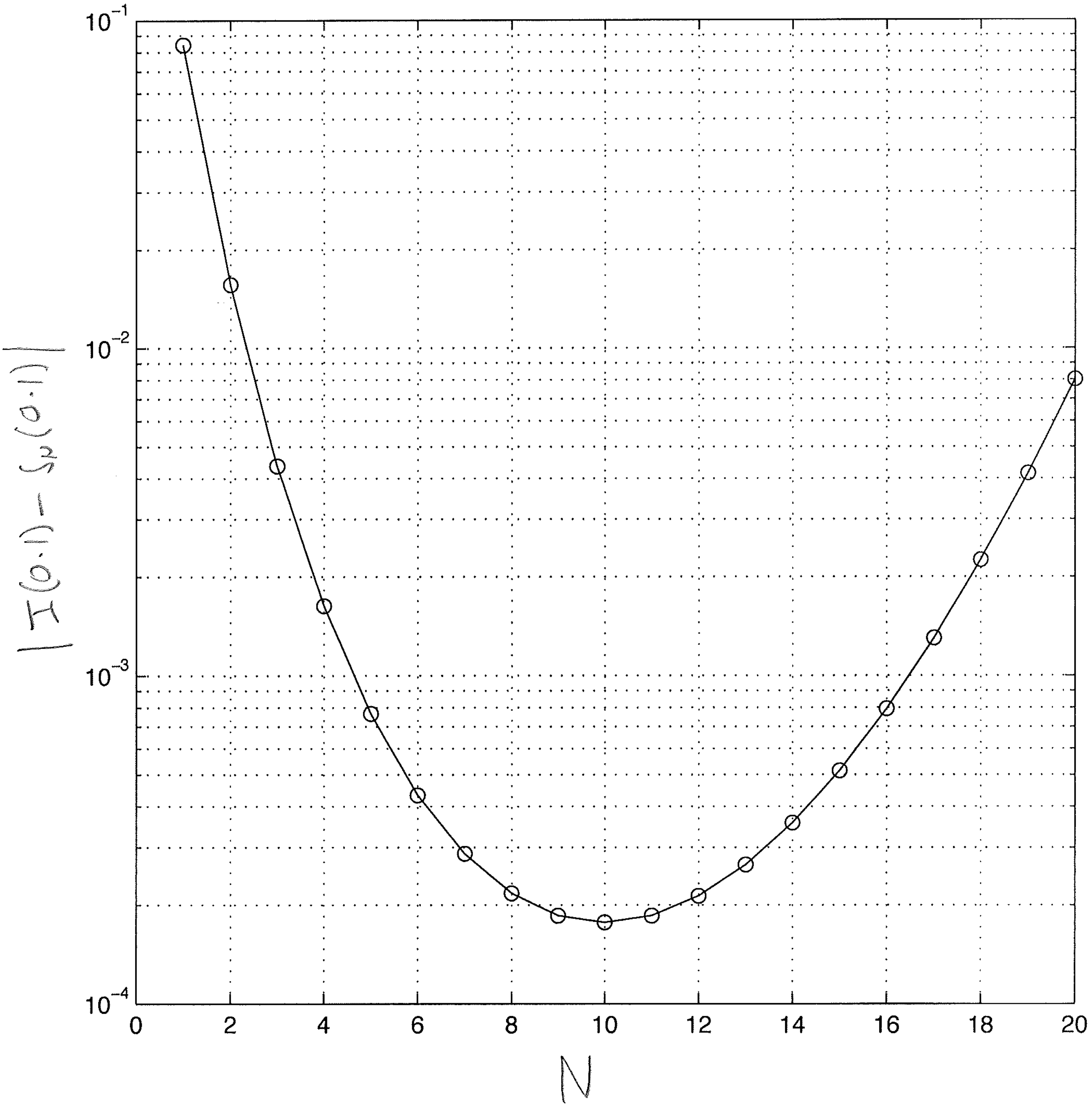
$\Rightarrow$ error exponentially small with optimal truncation!

$\boxed{\varepsilon = 0.2}$



(11)

$\varepsilon = 0.1$



Q5(a) $$I = \int_x^\infty t^\alpha e^{-t^\beta}\,dt$$

$$= \int_x^\infty \underbrace{\left(-\beta t^{\beta-1} e^{-t^\beta}\right)}_{f'} \underbrace{\left(-\frac{1}{\beta} t^{\alpha-\beta+1}\right)}_{g} dt$$

$$= f(t)g(t)\Big|_x^\infty - \int_x^\infty f(t)g'(t)\,dt \qquad (f(t) = e^{-t^\beta})$$

$$= \frac{1}{\beta} x^{\alpha-\beta+1} e^{-x^\beta} + \underbrace{\frac{\alpha-\beta+1}{\beta}}_{\neq 0} \underbrace{\int_x^\infty t^{\alpha-\beta} e^{-t^\beta} dt}_{I_1} \qquad (\text{all for } x>0)$$

For $x > 0$, $t^{-\beta} < x^{-\beta}$ for $t > x$, so

$$0 < I_1 \leq \frac{1}{x^\beta}\int_x^\infty t^\alpha e^{-t^\beta} dt = \frac{I}{x^\beta} \quad \text{for } x > 0$$

$$\Rightarrow |I_1| << |I| \text{ as } x \to \infty \Rightarrow \underline{\underline{I \sim \frac{1}{\beta} x^{\alpha-\beta+1} e^{-x^\beta} \text{ as } x \to \infty}} \quad \square$$

(b) $$J = \int_{x^\gamma}^\infty e^{-xt^3} dt \underset{t = x^{-1/3}s}{=} \int_{x^{\gamma+1/3}}^\infty e^{-s^3} \frac{ds}{x^{1/3}} \qquad (\text{for } x>0)$$

(i) $\gamma > -1/3 \Rightarrow x^{\gamma+1/3} \to \infty$ as $x \to \infty$, so apply part (a) with $x \mapsto x^{\gamma+1/3}$, $\alpha = 0$ and $\beta = 3$ to obtain

$$J \sim \frac{1}{3} x^{(\gamma+1/3)(0-3+1)} e^{-x^{(\gamma+1/3)3}} \cdot \frac{1}{x^{1/3}} \quad \text{as } x \to \infty$$

$$\Rightarrow \underline{\underline{J \sim \frac{1}{3x^{2\gamma+1}} \exp\left(-x^{3\gamma+1}\right) \text{ as } x \to \infty}}$$

(ii) $\gamma < -\frac{1}{3} \Rightarrow x^{\gamma+1/3} \to 0^+$ as $x \to \infty$, so use hint to obtain

$$J = \frac{1}{x^{1/3}}\Big[\Gamma\left(\tfrac{4}{3}\right) - \underbrace{\int_0^{x^{\gamma+1/3}} e^{-s^3} ds}_{= O(x^{\gamma+1/3}) \text{ as } x\to\infty}\Big] \qquad (\text{for } x>0)$$

$$\Rightarrow \underline{\underline{J \sim \frac{\Gamma(4/3)}{x^{1/3}} \text{ as } x \to \infty}}$$

Q6(a) $\displaystyle\int_0^x e^{t^3}\,dt = \int_0^x \underbrace{\frac{1}{3t^2}}_{f}\,\underbrace{3t^2e^{t^3}}_{g'}\,dt$

$$= \underbrace{\frac{1}{3t^2}}_{f}\,\underbrace{e^{t^3}}_{g}\Big|_0^x - \int_0^x \underbrace{\frac{-2}{3t^3}}_{f'}\,\underbrace{e^{t^3}}_{g}\,dt$$

$$= \infty$$

$\Rightarrow$ naive IBP fails

$\Rightarrow$ first split range of integration, e.g. $\int_0^x = \int_0^a + \int_a^x$ $(a>0)$

and use IBP on $\int_a^x$, which will work because contribution from the lower limit is exponentially small in comparison to that from upper limit.

Working with e^{t^3} a bit clunky, so let $s = t^3$, then

$$I(x) = \int_0^x e^{t^3}\,dt = \frac{1}{3}\int_0^{x^3} s^{-2/3}e^s\,ds$$

Let $\displaystyle J_n(x) = \int_1^{x^3} \underbrace{s^{-n}}_{f}\,\underbrace{e^s}_{g'}\,ds$

$$= \underbrace{s^{-n}}_{f}\,\underbrace{e^s}_{g}\Big|_1^{x^3} - \int_1^{x^3}\underbrace{-ns^{-(n+1)}}_{f'}\,\underbrace{e^s}_{g}\,ds$$

$$= \frac{e^{x^3}}{x^{3n}} - e + n\,J_{n+1}(x)$$

$$\Rightarrow \quad J_{2/3}(x) = \frac{e^{x^3}}{x^2} - e + \frac{2}{3}J_{5/3}(x)$$

$$= \frac{e^{x^3}}{x^2} - e + \frac{2}{3}\left[\frac{e^{x^3}}{x^5} - e + \frac{5}{3}J_{8/3}(x)\right]$$

$$= \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} - \frac{5}{3}e + \frac{10}{9}\left[J_{8/3}(x) - e + \frac{8}{3}J_{11/3}(x)\right]$$

$$\Rightarrow J_{2/3}(x) = \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} + \frac{10e^{x^3}}{9x^8} - \frac{25}{9}e + \frac{80}{27} J_{11/3}(x)$$

where
$$|J_{11/3}(x)| = \left| \int_1^{x^3} \frac{e^s}{s^{11/3}}\, ds \right|$$
$$< \left.\frac{e^s}{s^{11/3}}\right|_{s=x^3} \cdot \left| \int_1^{x^3} ds \right|$$
$$= \frac{e^{x^3}}{x^8} \quad \text{as } x \to \infty$$

Hence,

$$I(x) = \frac{1}{3}\int_0^1 s^{-2/3} e^s\, ds + \frac{1}{3} J_{2/3}(x)$$

$$\Rightarrow I(x) = \frac{e^{x^3}}{3x^2} + \frac{2e^{x^3}}{9x^5} + O\left(\frac{e^{x^3}}{x^8}\right) \quad \text{as } x \to \infty$$

(6)
$$I(x) = \int_0^\infty \underbrace{te^{-t^2}}_{f} \underbrace{\cos(xt)}_{g'}\, dt$$
$$= \underbrace{te^{-t^2}}_{f} \underbrace{\frac{\sin(xt)}{x}}_{g} \Big|_0^\infty - \int_0^\infty \underbrace{(1-2t^2)e^{-t^2}}_{f'} \underbrace{\frac{\sin(xt)}{x}}_{g}\, dt$$
$$= -\frac{1}{x}\int_0^\infty \underbrace{(1-2t^2)e^{-t^2}}_{f} \underbrace{\sin(xt)}_{g'}\, dt$$
$$= -\frac{1}{x}\left[\underbrace{(1-2t^2)e^{-t^2}}_{f} \underbrace{\frac{-\cos(xt)}{x}}_{g} - \int_0 \underbrace{(-4t-2t(1-2t^2))e^{-t^2}}_{f'} \underbrace{\frac{-\cos(xt)}{x}}_{g}\, dt \right]$$
$$= -\frac{1}{x^2} + \frac{1}{x^2}\int_0^\infty \underbrace{(6t-4t^3)e^{-t^2}}_{f} \underbrace{\cos(xt)}_{g'}\, dt$$
$$= -\frac{1}{x^2} + \frac{1}{x^2}\left[\underbrace{(6t-4t^3)e^{-t^2}}_{f} \underbrace{\frac{\sin(xt)}{x}}_{g} \Big|_0^\infty - \int_0^\infty \underbrace{(6-12t^2-2t(6t-4t^3))e^{-t^2}}_{f'} \underbrace{\frac{\sin(xt)}{x}}_{g}\, dt \right]$$
$$= -\frac{1}{x^2} + R(x), \text{ where}$$

$$|R(x)| = \frac{1}{x^3}\left| \int_0^\infty (6-24t^2+8t^4)e^{-t^2}\sin(xt)\, dt \right| \leq \frac{C}{x^3}, \text{ where}$$
$$C = \int_0^\infty |(6-24t^2+8t^4)e^{-t^2}|\, dt \Rightarrow I(x) = -\frac{1}{x^2} + O\left(\frac{1}{x^3}\right) \text{ as } x \to \infty$$