Q1

$\int_0^{\pi/2} e^{ix\cos t}\,dt$: stationary phase for first term;
steepest descents for more.

$\int_0^1 \ln t\, e^{ixt}\,dt$: steepest descents (IBP & stationary phase don't work)

$\int_0^x t^{-1/2} e^{-t}\,dt$: $\int_0^\infty - \int_x^\infty$ and IBP on $\int_x^\infty$.

$\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$: Laplace's method.

$\int_0^1 e^{ixe^{-1/t}}\,dt$: steepest descents having set $s = e^{-1/t}$.

$\int_0^{10} \frac{e^{-xt}}{1+t}\,dt$: Taylor expand integrand & integrate term-by-term

$\int_0^{\pi/2} \frac{dt}{\sqrt{\cos^2 t + x\sin^2 t}}$: $\int_0^{\pi/2} = \int_0^{\pi/2-\delta} + \int_{\pi/2-\delta}^{\pi}$, where $x \ll \delta \ll 1$.

$\int_0^1 \frac{\sin(xt)}{t}\,dt$: Taylor expand integrand & integrate term-by-term.

$\int_x^\infty t^{a-1} e^{-t}\,dt$: $\int_x^\infty = \int_0^\infty - \int_0^x$ and T on $\int_0^x$ for $\mathrm{Re}(a) > 0$;

much more tricky for $\mathrm{Re}(a) \le 0$!

$\int_0^1 \frac{\ln t}{x+t}\,dt$: $\int_0^1 = \int_0^\delta + \int_\delta^1$, where $x \ll \delta \ll 1$.

Q2

$$I_1(x) = \int_{-1}^{1} e^{-x\cosh t}\,dt \quad \text{as } x \to \infty.$$

Maximum of $\phi(t) = -\cosh t$ at $t=0 \Rightarrow$ split integral

$$I_1(x) = \Big(\underbrace{\int_{-1}^{-\varepsilon}}_{I_{11}(x)} + \underbrace{\int_{-\varepsilon}^{\varepsilon}}_{I_{12}(x)} + \underbrace{\int_{\varepsilon}^{1}}_{I_{11}(x)\text{ by symmetry}}\Big) e^{-x\cosh t}\,dt, \text{ where } \varepsilon \ll 1$$

$$I_{11}(x) = \int_{\varepsilon}^{1} e^{-x\cosh t}\,dt = O\left(e^{-x\cosh\varepsilon}\right) = O\left(e^{-x}.e^{-x\varepsilon^2/2}\right)$$

$$I_{12}(x) = \int_{-\varepsilon}^{\varepsilon} e^{-x[1+t^2/2+O(t^4)]}\,dt \qquad (\text{OK} \because \varepsilon \ll 1)$$

$$= e^{-x}\int_{-\varepsilon}^{\varepsilon} e^{-xt^2/2} e^{O(xt^4)}\,dt$$

$$= e^{-x}\int_{-\varepsilon}^{\varepsilon} e^{-xt^2/2}\left(1+O(xt^4)\right)dt \qquad (\text{provided } x\varepsilon^4 \ll 1)$$

$$= \sqrt{\tfrac{2}{x}}\, e^{-x}\int_{-\varepsilon\sqrt{\frac{x}{2}}}^{\varepsilon\sqrt{\frac{x}{2}}} e^{-s^2}\left(1+O(s^4/x)\right)ds \qquad \left(s=\sqrt{\tfrac{x}{2}}\,t\right)$$

$$= \sqrt{\tfrac{2}{x}}\, e^{-x}\left[\int_{-\varepsilon\sqrt{\frac{x}{2}}}^{\varepsilon\sqrt{\frac{x}{2}}} e^{-s^2}\,ds + O(x^{-1})\right]$$

(exp. small error for $\varepsilon\sqrt{x} \gg 1$)

$$= \sqrt{\tfrac{2}{x}}\, e^{-x}\left[\int_{-\infty}^{\infty} e^{-s^2}\,ds + O(x^{-1})\right] \qquad (\text{provided } \varepsilon\sqrt{x} \gg 1)$$

$$= \sqrt{\tfrac{2}{x}}\, e^{-x}\left[\sqrt{\pi} + O(x^{-1})\right] \qquad \text{as } x \to \infty$$

$\varepsilon\sqrt{x} \gg 1 \Rightarrow I_{11}(x) \ll I_{12}(x)$ as $x \to \infty$

Expansions self consistent provided $x^{-1/2} \ll \varepsilon \ll x^{-1/4}$ and therefore

$$\underline{\underline{I_1(x) \sim \sqrt{\tfrac{2\pi}{x}}\, e^{-x} \quad \text{as } x \to \infty}}$$

$$I_2(x) = \int_{-\pi/2}^{\pi/2} e^{-x(t^2-\sin^2 t)}\,dt \quad \text{as } x \to \infty.$$

$\phi(t) = \sin^2 t - t^2 < 0$ for $t \neq 0$, with $\phi(0) = 0$, so interior maximum at $t=0$.

$\phi(t) = \left(t - \frac{t^3}{3} + O(t^5)\right)^2 - t^2 = -\frac{1}{3}t^4 + O(t^6)$ as $t \to 0$, so this is a degenerate case.

Split: $$I_2(x) = \Big(\underbrace{\int_{-\pi/2}^{-\varepsilon}}_{I_{21}(x)} + \underbrace{\int_{-\varepsilon}^{\varepsilon}}_{I_{22}(x)} + \underbrace{\int_{\varepsilon}^{\pi/2}}_{I_{21}(x) \text{ by symmetry}}\Big) e^{x\phi(t)}\,dt, \quad \varepsilon \ll 1$$

$$I_{21}(x) = \int_{\varepsilon}^{\pi/2} e^{x\phi(t)}\,dt = O\left(e^{x\phi(\varepsilon)}\right) = O\left(e^{-x\varepsilon^4}\right)$$

$$I_{22}(x) = \int_{-\varepsilon}^{\varepsilon} e^{-x(t^4/3 + O(t^6))}\,dt \qquad (\text{OK} \because \varepsilon \ll 1)$$

$$= \int_{-\varepsilon}^{\varepsilon} e^{-xt^4/3}\left(1 + O(xt^6)\right)dt \qquad (\text{provided } x\varepsilon^6 \ll 1)$$

$$= \left(\frac{3}{x}\right)^{1/4} \int_{-(\frac{x}{3})^{1/4}\varepsilon}^{(\frac{x}{3})^{1/4}\varepsilon} e^{-s^4}\left(1 + O(x^{-1/2}s^6)\right)ds \qquad \left(s = \left(\tfrac{x}{3}\right)^{1/4} t\right)$$

(exp. small error for $x^{1/4}\varepsilon \gg 1$)

$$= \left(\frac{3}{x}\right)^{1/4}\left[\int_{-\infty}^{\infty} e^{-s^4}\,ds + O(x^{-1/2})\right] \qquad (\text{provided } x^{1/4}\varepsilon \gg 1)$$

$$= \left(\frac{3}{x}\right)^{1/4}\left[2\,\frac{\Gamma(1/4)}{4} + O(x^{-1/2})\right] \qquad (\text{by hint})$$

$x^{1/4}\varepsilon \gg 1 \Rightarrow I_{21}(x) \ll I_{22}(x)$ as $x \to \infty$

Expansions self-consistent provided $x^{-1/4} \ll \varepsilon \ll x^{-1/6}$ and therefore

$$\underline{\underline{I_2(x) \sim \frac{\Gamma(1/4)}{2}\left(\frac{3}{x}\right)^{1/4} \text{ as } x \to \infty}}$$

$$I_3(x) = \int_0^\infty e^{-2t - x/t^2}\,dt \quad \text{as } x \to \infty$$

$$\frac{\partial}{\partial t}\left(2t + \frac{x}{t^2}\right) = 2\left(1 - \frac{x}{t^3}\right) = 0 \text{ when } t = x^{1/3}$$

$\Rightarrow$ this is a moveable maximum problem.

Let $y = x^{1/3}$, $t = x^{1/3}u$, then $I_3(x) = y\int_0^\infty e^{-y(2u+1/u^2)}\,du$

$$\phi(u) = -2u - \frac{1}{u^2} \Rightarrow \phi'(u) = 2\left(\frac{1}{u^3} - 1\right) = 0 \text{ when } u = 1$$

$\phi(u) = \phi(1) + \frac{1}{2!}\phi''(1)(u-1)^2 + O((u-1)^3)$ as $u \to 1$, where $\phi(1) = -3$, $\phi''(1) = -6$.

Split: $$I_3(x) = \Bigg(\underbrace{\int_0^{1-\varepsilon}}_{I_{31}(y)} + \underbrace{\int_{1-\varepsilon}^{1+\varepsilon}}_{I_{32}(y)} + \underbrace{\int_{1+\varepsilon}^{\infty}}_{I_{33}(y)}\Bigg)\, y e^{y\phi(u)}\,du, \quad \varepsilon \ll 1$$

$$I_{31}(y) = O(y e^{y\phi(1-\varepsilon)}) = O(y e^{-3y} e^{-3y\varepsilon^2})$$

$$I_{33}(y) = O(y e^{y\phi(1+\varepsilon)}) = O(y e^{-3y} e^{-3y\varepsilon^2})$$

$$I_{32}(y) = \int_{1-\varepsilon}^{1+\varepsilon} y e^{y(-3-3(u-1)^2 + O((u-1)^3))}\,du \quad (\text{OK} \because \varepsilon \ll 1)$$

$$= y e^{-3y} \int_{1-\varepsilon}^{1+\varepsilon} e^{-3y(u-1)^2}\left(1 + O(y(u-1)^3)\right)du \quad (\text{provided } y\varepsilon^3 \ll 1)$$

$$= y e^{-3y} \frac{1}{\sqrt{3y}} \int_{-\varepsilon\sqrt{3y}}^{\varepsilon\sqrt{3y}} e^{-s^2}\left(1 + O(y^{-1/2}s^3)\right)ds \quad (\sqrt{3y}(u-1) = s)$$

$$= \sqrt{\frac{y}{3}}\, e^{-3y}\left[\int_{-\infty}^{\infty} e^{-s^2}\,ds + O(y^{-1/2})\right] \quad (\text{provided } \varepsilon\sqrt{y} \gg 1)$$

$$\varepsilon\sqrt{y} \gg 1 \Rightarrow I_{31}(y),\ I_{33}(y) \ll I_{32}(y) \text{ as } y \to \infty$$

Expansions self-consistent provided $x^{-1/6} \ll \varepsilon \ll x^{-1/9}$ and therefore

$$\underline{\underline{I_3(x) \sim \sqrt{\frac{\pi}{3}}\, x^{1/6} e^{-3x^{1/3}} \quad \text{as } x \to \infty}}$$

Q3

$$J_1(x) = \int_0^1 \cosh(t^2) e^{ixt^2} dt \quad \text{as } x \to \infty.$$

$\psi(t) = t^2$ has a stationary point at $t=0$, so split

$$J_1(x) = \Big(\underbrace{\int_0^\varepsilon}_{J_{11}(x)} + \underbrace{\int_\varepsilon^1}_{J_{12}(x)} \Big) \cosh(t^2) e^{ixt^2} dt, \qquad \varepsilon \ll 1$$

$$J_{11}(x) = \int_0^\varepsilon [1 + O(t^4)] e^{ixt^2} dt \qquad (\text{OK} \because \varepsilon \ll 1)$$

$$= \frac{1}{\sqrt{x}} \int_0^{\varepsilon\sqrt{x}} [1 + O(\tfrac{s^4}{x^2})] e^{is^2} ds \qquad (s = \sqrt{x}\, t)$$

$$= \frac{1}{\sqrt{x}} \int_0^\infty e^{is^2} ds + O\left(\frac{1}{\varepsilon x}\right) \qquad (\text{provided } \varepsilon\sqrt{x} \gg 1)$$

O(1/(sqrt(x) . eps . sqrt(x))

the last equality because

$$\frac{1}{\sqrt{x}} \int_{\varepsilon\sqrt{x}}^\infty e^{is^2} ds = \frac{1}{\sqrt{x}} \int_{\varepsilon\sqrt{x}}^\infty \underbrace{\frac{1}{2is}}_{f} \underbrace{2is e^{is^2}}_{g'} ds$$

$$= \frac{1}{\sqrt{x}} \left[\underbrace{\frac{1}{2is}}_{f} \underbrace{e^{is^2}}_{g} \Big|_{\varepsilon\sqrt{x}}^\infty - \int_{\varepsilon\sqrt{x}}^\infty \underbrace{\frac{-1}{2is^2}}_{f'} \underbrace{e^{is^2}}_{g} ds \right]$$

$$= \underbrace{-\frac{1}{2i\varepsilon x} e^{i\varepsilon^2 x}}_{O(\frac{1}{\varepsilon x})} + \underbrace{\frac{1}{2i\sqrt{x}} \int_{\varepsilon\sqrt{x}}^\infty \frac{1}{s^2} e^{is^2} ds}_{O(\frac{1}{\varepsilon x})}$$

and $\displaystyle \frac{1}{x^{5/2}} \int_0^{\varepsilon\sqrt{x}} s^4 e^{is^2} ds = O\left(\frac{(\varepsilon\sqrt{x})^3}{x^{5/2}}\right) = O\left(\frac{\varepsilon^3}{x}\right) \ll 1$ (provided $\varepsilon^3 \ll x$)

By parts ... first term

$$J_{12}(x) = \int_\varepsilon^1 \underbrace{\frac{\cosh(t^2)}{2ixt}}_{f} \underbrace{2ixt e^{ixt^2}}_{g'} dt$$

$$= \underbrace{\frac{\cosh(t^2)}{2ixt}}_{f} \underbrace{e^{ixt^2}}_{g} \Big|_\varepsilon^1 - \underbrace{\int_\varepsilon^1 \underbrace{\frac{\partial}{\partial t}\left(\frac{\cosh(t^2)}{2ixt}\right)}_{f'} \underbrace{e^{ixt^2}}_{g} dt}_{o(1/x) \text{ as } x \to \infty \text{ by RLL}}$$

$$= O\left(\frac{1}{\varepsilon x}\right)$$

Expansions selfconsistent provided $x^{-1/2} \ll \varepsilon \ll x^{-1/3}$ $(\Rightarrow x^{-1/2} \gg \frac{1}{\varepsilon x})$ and therefore (by hint)

$$J_1(x) \sim \frac{e^{i\pi/4}}{2} \sqrt{\frac{\pi}{x}} \quad \text{as } x \to \infty$$

$$J_2(x) = \mathrm{Re}\left[J_4(x) = \int_0^1 \tan(t) e^{ixt^4}\,dt\right] \text{ as } x \to \infty$$

$\psi(t) = t^4$ has a stationary point at $t = 0$, so split

$$J_4(x) = \Big(\underbrace{\int_0^\varepsilon}_{J_{41}(x)} + \underbrace{\int_\varepsilon^1}_{J_{42}(x)}\Big)\tan(t)\, e^{ixt^4}\,dt, \quad \varepsilon \ll 1.$$

$$J_{41}(x) = \int_0^\varepsilon \left(t + O(t^3)\right) e^{ixt^4}\,dt \qquad (\text{OK} \because \varepsilon \ll 1)$$

$$= x^{-1/4}\int_0^{x^{1/4}\varepsilon}\left(\frac{s}{x^{1/4}} + O\left(\frac{s^3}{x^{3/4}}\right)\right)e^{is^4}\,ds \qquad (s = x^{1/4}t)$$

$$= x^{-1/2}\int_0^\infty s e^{is^4}\,ds + O\left(\frac{1}{\varepsilon^2 x}\right) \qquad (\text{provided } x^{1/4}\varepsilon \gg 1)$$

need 1/(eps^2 x^1/2) << 1

the last equality because

$$x^{-1/2}\int_{x^{1/4}\varepsilon}^\infty s e^{is^4}\,ds = x^{-1/2}\int_{x^{1/4}\varepsilon}^\infty \underbrace{\frac{+1}{4is^2}}_{f}\ \underbrace{+4is^3 e^{is^4}}_{g'}\,ds$$

$$= x^{-1/2}\Bigg[\underbrace{\underbrace{\frac{1}{4is^2}}_{f}\ \underbrace{e^{is^4}}_{g}\Big|_{x^{1/4}\varepsilon}^\infty}_{O\left(\frac{1}{x^{1/2}\varepsilon^2}\right)} - \underbrace{\int_{x^{1/4}\varepsilon}^\infty \underbrace{\frac{-2}{4is^3}}_{f'}\ \underbrace{e^{is^4}}_{g}\,ds}_{O\left(\frac{1}{x^{1/2}\varepsilon^2}\right)}\Bigg]$$

$$= O\left(\frac{1}{x\varepsilon^2}\right)$$

and $x^{-1}\int_0^{x^{1/4}\varepsilon} s^3 e^{is^4}\,ds = \frac{1}{4ix} e^{is^4}\Big|_0^{x^{1/4}\varepsilon} = O\left(\frac{1}{x}\right)$

$$J_{42}(x) = \int_\varepsilon^1 \underbrace{\frac{\tan(t)}{4ixt^3}}_{f}\ \underbrace{4ixt^3 e^{ixt^4}}_{g'}\,dt$$

$$= \underbrace{\frac{\tan(t)}{4ixt^3}}_{f}\ \underbrace{e^{ixt^4}}_{g}\Big|_\varepsilon^1 - \underbrace{\int_\varepsilon^1 \underbrace{\frac{\partial}{\partial t}\left(\frac{\tan(t)}{4ixt^3}\right)}_{f'}\ \underbrace{e^{ixt^4}}_{g}\,dt}_{o(1/x) \text{ as } x \to \infty \text{ by RLL}}$$

$$= O\left(\frac{1}{x\varepsilon^2}\right)$$

Expansions self consistent provided $\varepsilon \gg x^{-1/4}$ $\left(\Rightarrow x^{-1/2} \gg \frac{1}{\varepsilon^2 x}\right)$
and therefore (by hint)

$$J_2(x) \sim \frac{\cos\frac{\pi}{4}}{4}\Gamma\left(\frac{1}{4}\right)x^{-1/2} \text{ as } x \to \infty$$

Gamma(1/2), typo in hint

$$J_3(x) = \int_0^1 e^{ix\psi(t)}\,dt \quad \text{as } x \to \infty$$

$$\psi(t) = t - \sin t = t - \left[t - \frac{t^3}{3!} + O(t^5)\right] = \frac{t^3}{6} + O(t^5) \text{ as } t \to 0$$

and $\psi'(t) = 1 - \cos t > 0$ for $0 < t \le 1$, so $t = 0$ is only stationary point and we split

$$J_3(x) = \Big(\underbrace{\int_0^\varepsilon}_{J_{31}(x)} + \underbrace{\int_\varepsilon^1}_{J_{32}(x)}\Big) e^{ix\psi(t)}\,dt, \quad \varepsilon \ll 1.$$

$$J_{31}(x) = \int_0^\varepsilon e^{ix[t^3/6 + O(t^5)]}\,dt \qquad (\text{OK} \because \varepsilon \ll 1)$$

$$= \left(\frac{6}{x}\right)^{1/3} \int_0^{\varepsilon(\frac{x}{6})^{1/3}} e^{is^3}\left(1 + O\left(x\left(\frac{s}{x^{1/3}}\right)^5\right)\right)ds \quad \left(s = \left(\tfrac{x}{6}\right)^{1/3} t \text{ with } x\varepsilon^5 \ll 1\right)$$

$$= \left(\frac{6}{x}\right)^{1/3} \int_0^\infty e^{is^3}\,ds + O\left(\frac{1}{\varepsilon^2 x}\right) \quad (\text{provided } x^{1/3}\varepsilon \gg 1)$$

the last equality because

$$x^{-1/3}\int_{\varepsilon(\frac{x}{6})^{1/3}}^{\infty} e^{is^3}\,ds = x^{-1/3}\int_{\varepsilon(\frac{x}{6})^{1/3}}^{\infty} \underbrace{\frac{1}{3is^2}}_{f}\ \underbrace{3is^2 e^{is^3}}_{g'}\,ds$$

$$= x^{-1/3}\left[\underbrace{\frac{1}{3is^2}}_{f}\ \underbrace{e^{is^3}}_{g}\Big|_{\varepsilon(\frac{x}{6})^{1/3}}^{\infty} - \int_{\varepsilon(\frac{x}{6})^{1/3}}^{\infty} \underbrace{\frac{\partial}{\partial s}\left(\frac{1}{3is^2}\right)}_{f'} \underbrace{e^{is^3}}_{g}\,ds\right]$$

$$= O\left(\frac{1}{\varepsilon^2 x}\right)$$

and
$$x^{-1}\int_0^{\varepsilon(\frac{x}{6})^{1/3}} s^5 e^{is^3}\,ds = x^{-1}\int_0^{\varepsilon(\frac{x}{6})^{1/3}} \underbrace{\frac{s^3}{3i}}_{f}\ \underbrace{3is^2 e^{is^3}}_{g'}\,ds$$

$$= x^{-1}\left[\underbrace{\frac{s^3}{3i}}_{f}\ \underbrace{e^{is^3}}_{g}\Big|_0^{\varepsilon(\frac{x}{6})^{1/3}} - \int_0^{\varepsilon(\frac{x}{6})^{1/3}} \underbrace{\frac{s^2}{i}}_{f'}\ \underbrace{e^{is^3}}_{g}\,ds\right]$$

$$= O(\varepsilon^3) + O(1/x)$$

Expansions self-consistent provided $x^{-1/3} \ll \varepsilon \ll x^{-1/5}$
and therefore (by hint)
$$\underline{\underline{J_3(x) \sim \left(\frac{2}{9}\right)^{1/3} \Gamma\left(\frac{1}{3}\right) e^{i\pi/6} x^{-1/3} \text{ as } x \to \infty}}$$

(7)

Q4

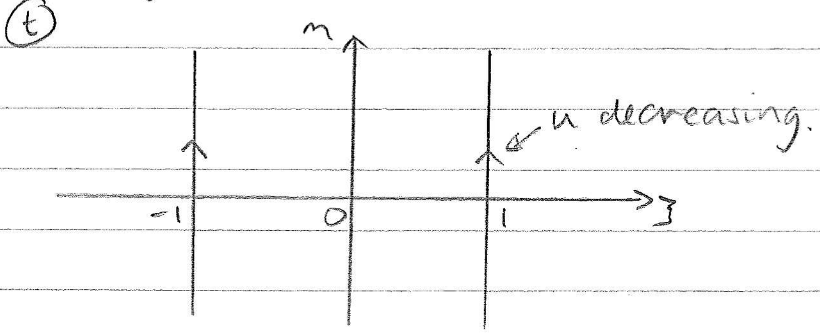
$$I(x) = \int_{-1}^{1} f(t) e^{x\phi(t)}\, dt, \text{ where } f(t) = (1-t^2)^N, \ \phi(t) = it \quad (N \in \mathbb{N})$$

(a) Let $\phi(t) = u(\xi, \eta) + i v(\xi, \eta)$, $t = \xi + i\eta$

$\Rightarrow \quad u = -\eta, \quad v = \xi$

On a steepest descent contour (SDC), the phase v is constant and $-\nabla u = -(u_\xi, u_\eta) = (0, 1)$ is a tangent pointing in the direction in which u decreases most rapidly.

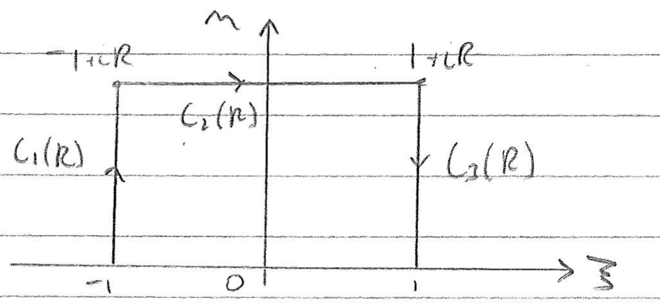
Thus, SDC through $t = \pm 1$ are $\xi = \pm 1$:



(b) Integrand a holomorphic function of t on $\mathbb{C}$, so by deformation theorem

$$I(x) = \left(\int_{C_1(R)} + \int_{C_2(R)} + \int_{C_3(R)} \right) f(t) e^{x\phi(t)}\, dt$$

where $C_i(R)$ are as shown:



Since $\left| f(t) e^{x\phi(t)} \right| = |f(t)|\, e^{-xR} \to 0$ as $R \to \infty$ on $C_2(R)$, let $R \to \infty$ to obtain

$$I(x) = \left(\int_{C_1(\infty)} + \int_{C_3(\infty)} \right) f(t) e^{x\phi(t)}\, dt = \left(\int_{-1}^{-1+i\infty} - \int_{1}^{1+i\infty} \right) f(t) e^{x\phi(t)}\, dt$$

i.e. $I(x) = I_-(x) - I_+(x)$, where $I_\pm(x) = \int_{\pm 1}^{\pm 1 + i\infty} f(t)\, e^{x\phi(t)}\, dt$

(c) $I_\pm(x) = i e^{\pm i x} \int_0^\infty f(\pm 1 + is)\, e^{-xs}\, ds \qquad (t = \pm 1 + is,\ s > 0)$

where $f(\pm 1 + is) = \left[1 - (\pm 1 + is)^2\right]^N = \begin{cases} (-is)^N (2+is)^N, & \oplus \\ (2+is)^N (is)^N, & \ominus \end{cases}$

Laplace's method:

$$I_\pm(x) = i e^{\pm i x} \left(\int_0^\varepsilon + \int_\varepsilon^\infty \right) f(\pm 1 + is)\, e^{-xs}\, ds, \quad \varepsilon \ll 1$$

$\oplus$ first:

$$\int_0^\varepsilon f(1+is)\, e^{-xs}\, ds = \int_0^\varepsilon (-is)^N (2+is)^N e^{-xs}\, ds$$

$$= (-2i)^N \int_0^\varepsilon \left(s^N + O(s^{N+1})\right) e^{-xs}\, ds \qquad (\because \varepsilon \ll 1)$$

$$= \frac{(-2i)^N}{x^{N+1}} \left[\int_0^{\varepsilon x} u^N e^{-u}\, du + O(x^{-1}) \right] \qquad \left(s = \frac{u}{x}\right)$$

$$= \frac{(-2i)^N}{x^{N+1}} \left[\underbrace{\int_0^\infty u^N e^{-u}\, du}_{\Gamma(N+1) = N!} + O(x^{-1}) \right] \qquad (\varepsilon x \gg 1)$$

$$= \frac{(-2i)^N N!}{x^{N+1}} + O\left(\frac{1}{x^{N+2}}\right)$$

$$\int_\varepsilon^\infty f(1+is)\, e^{-xs}\, ds = O(e^{-\varepsilon x})$$

Expansions self-consistent provided $x^{-1} \ll \varepsilon \ll 1$, so

$$I_+(x) \sim \frac{i e^{ix} (-2i)^N N!}{x^{N+1}} \quad \text{as } x \to \infty.$$

Similarly, $I_-(x) \sim \dfrac{i e^{-ix} (2i)^N N!}{x^{N+1}}$ as $x \to \infty$.

$$\Rightarrow \quad I(x) \sim \frac{2^N i^{N+1} N!}{x^{N+1}} \left(e^{-ix} - (-1)^N e^{ix} \right) \quad \text{as } x \to \infty$$

NB: real as required by symmetry of integrand!

Q5

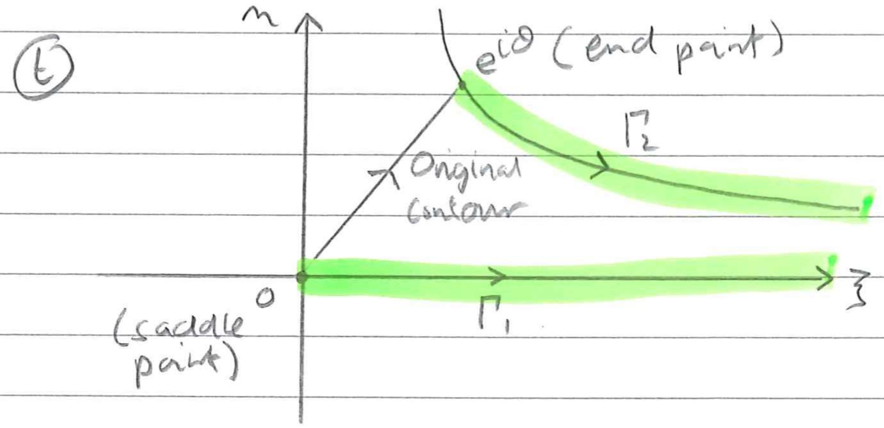
$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\,ds = \frac{2r}{\sqrt{\pi}} \int_0^{e^{i\theta}} e^{r^2\phi(t)}\,dt$$

where $z = re^{i\theta}$, $s = rt$ and $\phi(t) = -t^2$.

Let $\phi(t) = u(\xi,\eta) + iv(\xi,\eta)$, $t = \xi + i\eta$

$\Rightarrow \quad u = \eta^2 - \xi^2, \quad v = -2\xi\eta, \quad -\nabla u = 2(\xi, -\eta)$

$\Rightarrow$ • SDC through $t = 0$ is $\eta = 0$ (with $v = 0$)
• SDC through $t = e^{i\theta}$ ($0 < \theta < \pi/2$) is $2\xi\eta = 2\cos\theta\sin\theta = \sin 2\theta$, $\xi > 0, \eta > 0$ (with $v = -\sin 2\theta$).



Deformation theorem $\Rightarrow$ $\mathrm{erf}(z) = \Big(\underbrace{\int_{\Gamma_1}}_{I_1(r)} - \underbrace{\int_{\Gamma_2}}_{I_2(r,\theta)}\Big) \frac{2r}{\sqrt{\pi}} e^{r^2\phi(t)}\,dt$

$$I_1(r) = \frac{2r}{\sqrt{\pi}} \int_0^\infty e^{-r^2\xi^2}\,d\xi \underset{\xi = \hat{\xi}/r}{=} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\hat{\xi}^2}\,d\hat{\xi} = 1$$

$$I_2(r,\theta) = \frac{2r}{\sqrt{\pi}} \int_{\cos\theta}^\infty e^{r^2(u(\xi,\eta(\xi)) - i\sin 2\theta)} (1 + i\eta'(\xi))\,d\xi$$

because $\Gamma_2 = \{\xi + i\eta(\xi) : \xi > \cos\theta\}$, where $\eta(\xi) = \frac{\sin 2\theta}{2\xi}$

$$\Rightarrow I_2(r,\theta) = \frac{2r}{\sqrt{\pi}} e^{-r^2 i \sin 2\theta} \int_{\cos\theta}^\infty F(\xi)\, e^{r^2\Phi(\xi)}\,d\xi$$

where $F(\xi) = 1 + i\eta'(\xi) = 1 - \frac{i\sin 2\theta}{2\xi^2}$, $\Phi(\xi) = u(\xi,\eta(\xi)) = \frac{\sin^2 2\theta}{4\xi^2} - \xi^2$

Γ_2 a SDC $\Rightarrow \Phi(\xi) = u(\xi, \eta(\xi))$ decreasing with ξ on Γ_2

Hence, Laplace's method gives

$$I_2(r,\theta) \sim -\frac{2r}{\sqrt{\pi}} e^{-r^2 i \sin 2\theta} \frac{F(\cos\theta)\, e^{r^2 \Phi(\cos\theta)}}{r^2 \Phi'(\cos\theta)} \quad \text{as } r \to \infty$$

where $F(\cos\theta) = \dfrac{e^{-i\theta}}{\cos\theta}$, $\Phi(\cos\theta) = -\cos 2\theta$, $\Phi'(\cos\theta) = \dfrac{-2}{\cos\theta}$

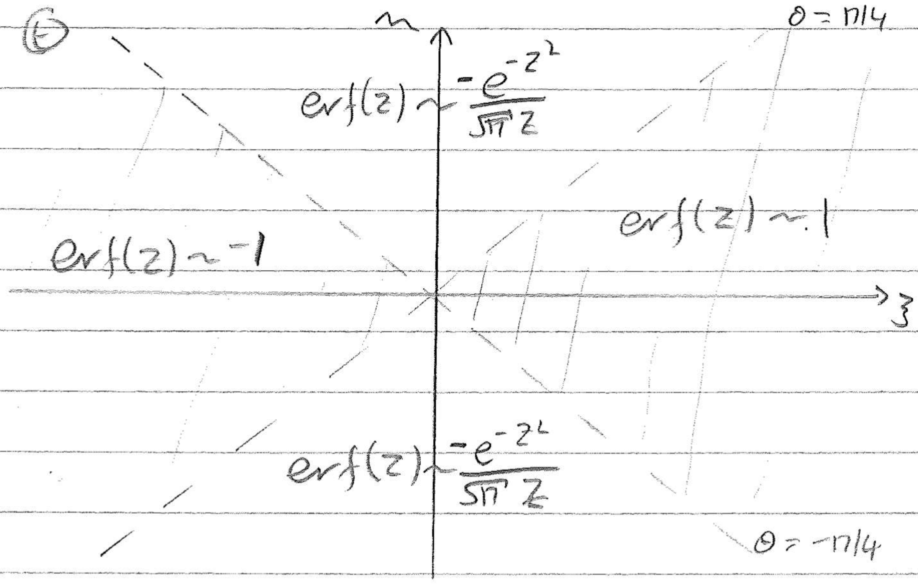
$$\Rightarrow I_2(r,\theta) \sim +\frac{1}{\sqrt{\pi}\, r e^{i\theta}} e^{-r^2 e^{2i\theta}} \quad \text{as } r \to \infty.$$

Thus, $I_1 \sim 1$ and $I_2 \sim +\frac{1}{\sqrt{\pi} z} e^{-z^2}$ as $r = |z| \to \infty$ for $0 < \theta = \arg(z) < \pi/2$

$$|I_2| \sim \frac{1}{r} e^{-r^2 \cos 2\theta} \begin{cases} \ll 1 & \text{for } 0 < \theta \le \pi/4 \\ \gg 1 & \text{for } \pi/4 < \theta < \pi/2 \end{cases}$$

$$\Rightarrow \operatorname{erf}(z) \sim \begin{cases} 1 & \text{for } 0 < \theta \le \pi/4 \\ -\frac{1}{\sqrt{\pi} z} e^{-z^2} & \text{for } \pi/4 < \theta < \pi/2 \end{cases} \quad \text{as } |z| \to \infty$$

NB: More generally



- Different asymptotic expansion in different sectors called Stokes phenomenon
- While e^{-z^2} is entire, it has an essential singularity at ∞
- $\theta = \pm \pi/2$ are Stokes lines (across which topology of SDC changes).
- $|\theta| = \frac{\pi}{4}, \frac{5\pi}{4}$ are anti-Stokes lines (across which dominance of end point and saddle point changes).

Q6

$$I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx$$

$$= \Big(\underbrace{\int_0^\delta}_{I_1} + \underbrace{\int_\delta^1}_{I_2}\Big) \frac{f(x)}{x+\varepsilon}\,dx \quad , \text{where } \varepsilon \ll \delta \ll 1.$$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{f(\varepsilon y)\,dy}{y+1} \qquad (x = \varepsilon y)$$

$$= \int_0^{\delta/\varepsilon} \frac{1}{y+1}\left(f(0) + \varepsilon y f'(0) + O(\varepsilon^2)\right) dy \qquad (\because \varepsilon y \ll \delta \ll$$

$$= f(0)\ln(y+1)\Big|_0^{\delta/\varepsilon} + O(\delta)$$

$$= f(0)\ln\left(1+\frac{\delta}{\varepsilon}\right) + O(\delta)$$

$$= f(0)\ln\left(\frac{\delta}{\varepsilon}\right) + f(0)\ln\left(1+\frac{\varepsilon}{\delta}\right) + O(\delta)$$

$$= -f(0)\ln\varepsilon + f(0)\ln\delta + O\left(\delta, \frac{\varepsilon}{\delta}\right)$$

$$I_2 = \int_\delta^1 \frac{f(x)\,dx}{x+\varepsilon}$$

$$= \int_\delta^1 \frac{f(x)}{x(1+\frac{\varepsilon}{x})}\,dx$$

$$= \int_\delta^1 \frac{f(x)}{x}\left(1 - \frac{\varepsilon}{x} + O(\varepsilon^2)\right) dx \qquad \left(\because \frac{\varepsilon}{x} < \frac{\varepsilon}{\delta} \ll 1\right)$$

$$\sim \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx + \int_\delta^1 \frac{f(0)}{x}\,dx + \cdots$$

$$\sim -f(0)\ln\delta + \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx + \cdots$$

$$\Rightarrow I \sim -f(0)\ln\varepsilon + f(0)\ln\delta - f(0)\ln\delta + \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx + \cdots$$

$$\sim -f(0)\ln\varepsilon + \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \cdots \quad \text{as } \varepsilon \to 0$$

Q1

$\int_0^{\pi/2} e^{ix\cos t}\,dt$: stationary phase for first term;
steepest descents for more.

$\int_0^1 \ln t\, e^{ixt}\,dt$: steepest descents (IBP & stationary phase don't work)

$\int_0^x t^{-1/2} e^{-t}\,dt$: $\int_0^\infty - \int_x^\infty$ and IBP on $\int_x^\infty$.

$\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$: Laplace's method.

$\int_0^1 e^{ixe^{-1/t}}\,dt$: steepest descents having set $s = e^{-1/t}$.

$\int_0^{10} \frac{e^{-xt}}{1+t}\,dt$: Taylor expand integrand & integrate term-by-term

$\int_0^{\pi/2} \frac{dt}{\sqrt{\cos^2 t + x\sin^2 t}}$: $\int_0^{\pi/2} = \int_0^{\pi/2-\delta} + \int_{\pi/2-\delta}^{\pi}$, where $x \ll \delta \ll 1$.

$\int_0^1 \frac{\sin(xt)}{t}\,dt$: Taylor expand integrand & integrate term-by-term.

$\int_x^\infty t^{a-1} e^{-t}\,dt$: $\int_x^\infty = \int_0^\infty - \int_0^x$ and T on $\int_0^x$ for $\mathrm{Re}(a) > 0$;

much more tricky for $\mathrm{Re}(a) \leq 0$!

$\int_0^1 \frac{\ln t}{x+t}\,dt$: $\int_0^1 = \int_0^\delta + \int_\delta^1$, where $x \ll \delta \ll 1$.