Perturbation Methods : Problem Sheet 3

Q1(a) Van Dyke's matching rule "(m.t.i.)(n.t.o.) = (n.t.o.)(m.t.i.)" says that n terms in the outer expansion, written in inner variables, and reexpanded to m terms, is the same as m terms in the inner expansion, written in outer variables, and reexpanded to n terms.

(b) $f(x,\varepsilon) = \left[1 + (x+\varepsilon)^{1/2}\right]^{1/2}$

$\varepsilon \to 0^+$ with $x = O(1) \Rightarrow$
$$\begin{aligned} f(x,\varepsilon) &= \left[1 + x^{1/2}(1+\varepsilon/x)^{1/2}\right]^{1/2} \\ &\sim \left[1 + x^{1/2}\left(1 + \tfrac{\varepsilon}{2x} + \cdots\right)\right]^{1/2} \\ &= \left[1 + x^{1/2} + \tfrac{\varepsilon}{2x^{1/2}} + \cdots\right]^{1/2} \\ &= (1+x^{1/2})^{1/2}\left[1 + \tfrac{\varepsilon}{2x^{1/2}(1+x^{1/2})} + \cdots\right]^{1/2} \\ &\sim (1+x^{1/2})^{1/2}\left[1 + \tfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})} + \cdots\right] \\ &= (1+x^{1/2})^{1/2} + \tfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}} + \cdots \end{aligned}$$

$\Rightarrow$ (1.t.o.) $= (1+x^{1/2})^{1/2}$, (2.t.o.) $= (1+x^{1/2})^{1/2} + \dfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$

$\varepsilon \to 0^+$ with $X = x/\varepsilon = O(1) \Rightarrow$
$$\begin{aligned} f(\varepsilon X,\varepsilon) &= \left[1 + (\varepsilon X + \varepsilon)^{1/2}\right]^{1/2} \\ &= \left[1 + \varepsilon^{1/2}(X+1)^{1/2}\right]^{1/2} \\ &\sim 1 + \tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2} + \cdots \end{aligned}$$

$\Rightarrow$ (1.t.i.) $= 1$, (2.t.i.) $= 1 + \frac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$

$(m,n) = (1,1)$: (1.t.o.) $= (1+x^{1/2})^{1/2}$

$\Rightarrow$ (1.t.o.) in inner variables $= (1+(\varepsilon X)^{1/2})^{1/2} \sim 1 + \frac{1}{2}\varepsilon^{1/2}X^{1/2}$

$\Rightarrow$ (1.t.i.)(1.t.o.) $= 1$

(1.t.i.) $= 1$

$\Rightarrow$ (1.t.i.) in outer variables $= 1$

$\Rightarrow$ (1.t.o.)(1.t.i.) $= 1$

i.e. (1.t.i.)(1.t.o.) = (1.t.o.)(1.t.i.)

<u>$(m,n)=(1,2)$:</u> (2.t.o.) $= (1+x^{1/2})^{1/2} + \dfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$

$\Rightarrow$ (2.t.o.) in inner variables $= (1+(\varepsilon X)^{1/2})^{1/2} + \dfrac{\varepsilon}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$

$\sim 1 + \frac{1}{2}\varepsilon^{1/2}X^{1/2} + \dfrac{\varepsilon^{1/2}}{4X^{1/2}} + \dots$

$\Rightarrow$ (1.t.i.)(2.t.o.) $= 1$

(1.t.i.) $= 1$

$\Rightarrow$ (1.t.i.) in outer variables $= 1$

$\Rightarrow$ (2.t.o.)(1.t.i.) $= 1$

i.e. (1.t.i.)(2.t.o.) = (2.t.o.)(1.t.i.)

<u>$(m,n)=(2,1)$:</u> (1.t.o.) $= (1+x^{1/2})^{1/2}$

$\Rightarrow$ (1.t.o.) in inner variables $= (1+(\varepsilon X)^{1/2})^{1/2} \sim 1 + \frac{1}{2}\varepsilon^{1/2}X^{\frac{1}{2}} + \dots$

$\Rightarrow$ (2.t.i.)(1.t.o.) $= 1 + \frac{1}{2}\varepsilon^{1/2}X^{1/2}$

(2.t.i.) $= 1 + \frac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$

$\Rightarrow$ (2.t.i.) in outer variables $= 1 + \frac{1}{2}\varepsilon^{1/2}\left(\frac{x}{\varepsilon}+1\right)^{1/2}$

$= 1 + \frac{1}{2}x^{1/2}\left(1+\frac{\varepsilon}{x}\right)^{1/2}$

$\sim 1 + \frac{1}{2}x^{1/2} + \dots$

$\Rightarrow$ (1.t.o.)(2.t.i.) $= 1 + \frac{1}{2}x^{1/2}$

i.e. (2.t.i.)(1.t.o.) = (1.t.o.)(2.t.i.)

<u>$(m,n)=(2,2)$:</u> (2.t.o.) $= (1+x^{1/2})^{1/2} + \dfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$

$\Rightarrow$ (2.t.o.) in inner variables $= (1+(\varepsilon X)^{1/2})^{1/2} + \dfrac{\varepsilon}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$

$\sim 1 + \frac{1}{2}\varepsilon^{1/2}X^{1/2} + \dfrac{\varepsilon^{1/2}}{4X^{1/2}} + \dots$

$\Rightarrow$ (2.t.i.)(2.t.o.) $= 1 + \varepsilon^{1/2}\left(\frac{1}{2}X^{1/2} + \dfrac{1}{4X^{1/2}}\right)$

(2.t.i.) $= 1 + \frac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$

$\Rightarrow$ (2.t.i.) in outer variables $= 1 + \frac{1}{2}\varepsilon^{1/2}\left(\frac{x}{\varepsilon}+1\right)^{1/2}$

$\sim 1 + \frac{1}{2}x^{1/2} + \dfrac{\varepsilon}{4x^{1/2}} + \dots$

$\Rightarrow$ (2.t.o.)(2.t.i.) $= 1 + \frac{1}{2}x^{1/2} + \dfrac{\varepsilon}{4x^{1/2}}$

i.e. (2.t.i.)(2.t.o.) = (2.t.o.)(2.t.i.)

(c) $g(x,\varepsilon) = 1 + \dfrac{\log x}{\log \varepsilon} \sim \begin{cases} 1 + \dfrac{\log x}{\log \varepsilon}, & \varepsilon \to 0^+ \text{ with } x = O(1) \\ 2 + \dfrac{\log X}{\log \varepsilon}, & \varepsilon \to 0^+ \text{ with } X = x/\varepsilon = O(1) \end{cases}$

$\Rightarrow \text{(1.t.o.)} = 1, \quad \text{(1.t.i.)} = 2$

$\Rightarrow \text{(1.t.i.)(1.t.o.)} = 1 \neq 2 = \text{(1.t.o.)(1.t.i.)}$

Resolve by treating $\log \varepsilon$ as $O(1)$ for purposes of matching

$\Rightarrow \text{(1.t.o.)} = 1 + \dfrac{\log x}{\log \varepsilon}, \quad \text{(1.t.i.)} = 2 + \dfrac{\log X}{\log \varepsilon}$

$\Rightarrow \text{(1.t.i.)(1.t.o.)} = 2 + \dfrac{\log X}{\log \varepsilon} = 1 + \dfrac{\log x}{\log \varepsilon} = \text{(1.t.o.)(1.t.i.)}$

Q2(a) $\varepsilon y' + y = x$ for $x > 0$, with $y(0) = 1$.

<u>Outer:</u> $y \sim y_0(x) + \varepsilon y_1(x) + \dots$ as $\varepsilon \to 0^+$ with $x = O(1)$.

$O(\varepsilon^0)$: $\underline{y_0 = x}$

$O(\varepsilon^1)$: $y_0' + y_1 = 0 \Rightarrow \underline{y_1 = -1}$

<u>Inner:</u> $y(x) = Y(X)$, $X = x/\varepsilon = O(1)$

$\Rightarrow \dfrac{dY}{dX} + Y = \varepsilon X$ for $X > 0$, with $Y(0) = 1$

$Y \sim Y_0(X) + \varepsilon Y_1(X) + \dots$ as $\varepsilon \to 0^+$ with $X = O(1)$.

$O(\varepsilon^0)$: $\dfrac{dY_0}{dX} + Y_0 = 0$, $Y_0(0) = 1 \Rightarrow \underline{Y_0 = e^{-X}}$

$O(\varepsilon^1)$: $\dfrac{dY_1}{dX} + Y_1 = X$, $Y_1(0) = 0 \Rightarrow \underline{Y_1 = e^{-X} + X - 1}$

<u>Matching:</u> $\text{(2.t.o.)} = x - \varepsilon$

$\Rightarrow$ (2.t.o.) in inner variables $= \varepsilon X - \varepsilon$

$\Rightarrow \text{(2.t.i.)(2.t.o.)} = \varepsilon(X - 1)$

$$(2.t.i.) = e^{-X} + \varepsilon(e^{-X} + X - 1)$$

$$\Rightarrow (2.t.i.) \text{ in outer variables } = e^{-x/\varepsilon} + \varepsilon(e^{-x/\varepsilon} + \tfrac{x}{\varepsilon} - 1)$$
$$\sim x - \varepsilon + \text{E.S.T.}$$

$$\Rightarrow (2.t.o.)(2.t.i.) = x - \varepsilon$$

Hence, $\underline{\underline{(2.t.i.)(2.t.o.) = (2.t.o.)(2.t.i.)}}$

NB: exact solution is $y = (1+\varepsilon)e^{-x/\varepsilon} + x - \varepsilon$.

(b) $(x+\varepsilon)y' + y = 0$ for $x > 0$, with $y(0) = 1$.

Outer: $y \sim y_0(x) + \varepsilon y_1(x) + \dots$ as $\varepsilon \to 0^+$ with $x = O(1)$.

$O(\varepsilon^0)$: $x y_0' + y_0 = 0 \Rightarrow \underline{\underline{y_0 = A_1/x}} \quad (A_1 \in \mathbb{R})$

$O(\varepsilon^1)$: $x y_1' + y_0' + y_1 = 0 \Rightarrow \underline{\underline{y_1 = -\dfrac{A_1}{x^2} + \dfrac{A_2}{x} \quad (A_2 \in \mathbb{R})}}$

Inner: $y(x) = Y(X)$, $X = \frac{x}{\varepsilon} = O(1)$

$$\Rightarrow (1+X)\frac{dY}{dX} + Y = 0 \text{ for } X > 0, \text{ with } Y(0) = 1$$

$$Y \sim Y_0(X) + \varepsilon Y_1(X) + \dots \text{ as } \varepsilon \to 0^+ \text{ with } X = O(1)$$

$O(\varepsilon^0)$: $(1+X)\dfrac{dY_0}{dX} + Y_0 = 0$, $Y_0(0) = 1 \Rightarrow \underline{\underline{Y_0 = \dfrac{1}{1+X}}}$

$O(\varepsilon^1)$: $(1+X)\dfrac{dY_1}{dX} + Y_1 = 0$, $Y_1(0) = 0 \Rightarrow \underline{\underline{Y_1 = 0}}$

Matching: $(2.t.i.) = \dfrac{1}{1+X}$

$$\Rightarrow (2.t.i.) \text{ in outer variables } = \frac{1}{1+x/\varepsilon} \sim \varepsilon/x$$

$$\Rightarrow (2.t.o.)(2.t.i.) = \varepsilon/x$$

eps/(x+eps)
=0+(eps/x)/(1+eps/x)
=0+(eps/x)(1-eps/x+...)
=0+eps/x -(eps/x)2 + ...

$$(2.t.o.) = A_1/x + \varepsilon\left(-\frac{A_1}{x^2} + \frac{A_2}{x}\right)$$

$$\Rightarrow (2.t.o.) \text{ in inner variables } = \frac{A_1}{\varepsilon X} + \varepsilon\left(-\frac{A_1}{\varepsilon^2 X^2} + \frac{A_2}{\varepsilon X}\right)$$
$$\sim \frac{1}{\varepsilon}\left(\frac{A_1}{X} - \frac{A_1}{X^2}\right) + \frac{A_2}{X}$$

$$\Rightarrow (2.t.i.)(2.t.o.) = \frac{1}{\varepsilon}\left(\frac{A_1}{X} - \frac{A_1}{X^2}\right) + \frac{A_2}{X}$$
$$= \frac{A_1}{x} + \varepsilon\left(-\frac{A_1}{x^2} + \frac{A_2}{x}\right)$$

$(2.t.o.)(2.t.i.) = (2.t.i.)(2.t.o.) \Rightarrow \underline{\underline{A_1 = 0,\ A_2 = 1}}.$

Hence, $\underline{\underline{y \sim \frac{\varepsilon}{x} \text{ for } x = O(1),\ y \sim \frac{1}{1+X} \text{ for } X = O(1) \text{ as } \varepsilon \to 0}}$

Q3 $\varepsilon y'' + x^{1/2} y' + y = 0$ for $0 < x < 1$, with $y(0) = 0, y(1) = 1$.

(a) $x = 1 + \delta(\varepsilon) X,\ y = Y(X)$ with $X = O(1)$, $\delta \to 0$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2}\frac{d^2Y}{dX^2} + \frac{(1+\delta X)^{1/2}}{\delta}\frac{dY}{dX} + Y = 0$$

Balance 1st and 2nd term by setting $\delta = \varepsilon$ to obtain

$$\frac{d^2Y}{dX^2} + (1+\varepsilon X)^{1/2}\frac{dY}{dX} + \varepsilon Y = 0$$

for $X < 0$, with $Y(0) = 1$ upon imposing BC at $x = 1$.

$Y \sim Y_0(X) + \varepsilon Y_1(X) + \cdots$ as $\varepsilon \to 0^+$ with $X = O(1)$

$O(\varepsilon^0)$: $\frac{d^2Y_0}{dX^2} + \frac{dY_0}{dX} = 0,\ Y_0(0) = 1 \Rightarrow Y_0 = A + (1-A)e^{-X}\ (A \in \mathbb{R})$

Matching will require $Y(-\infty)$ finite $\Rightarrow A = 1 \Rightarrow Y_0 \equiv 1$, i.e. no BL! □

(b) $y \sim y_0(x) + \varepsilon y_1(x) + \ldots$ as $\varepsilon \to 0^+$ with $x = O(1)$.

$O(\varepsilon^0)$: $x^{1/2} y_0' + y_0 = 0$, $y_0(1) = 1$

$$\Rightarrow \quad \frac{y_0'}{y_0} = -\frac{1}{x^{1/2}} \Rightarrow \ln|y_0| = -2x^{1/2} + C_1 \Rightarrow y_0 = C_2 e^{-2x^{1/2}}$$

where $C_1, C_2 \in \mathbb{R}$, with $|C_2| = e^{C_1}$.

$y_0(1) = 1 \Rightarrow C_2 e^{-2} = 1 \Rightarrow C_2 = e^2 \Rightarrow \underline{\underline{y_0 = e^{2(1-x^{1/2})}}}$

□

(c) $x = \delta(\varepsilon)X$, $y = Y(X)$ with $X = O(1)$, $\delta \to 0$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2}\frac{d^2Y}{dX^2} + \frac{(\delta X)^{1/2}}{\delta}\frac{dY}{dX} + Y = 0$$

3^{rd} term << 2^{nd} term, so balance 1^{st} and 2^{nd} term by setting $\frac{\varepsilon}{\delta^2} = \frac{\delta^{1/2}}{\delta} \Rightarrow \delta = \varepsilon^{2/3}$ and

$$\frac{d^2Y}{dX^2} + X^{\frac{1}{2}}\frac{dY}{dX} + \varepsilon^{1/3}Y = 0 \quad \text{for } X > 0$$

with $Y(0) = 0$.

NB: Thus, we should have expanded $y \sim y_0(x) + \varepsilon^{1/3}y_1(x) + \ldots$ as $\varepsilon \to 0^+$ with $x = O(1)$ in outer region.

$Y \sim Y_0(X) + \varepsilon^{1/3}Y_1(X) + \ldots$ as $\varepsilon \to 0^+$ with $X = O(1)$.

$O(\varepsilon^0)$: $\frac{d^2Y_0}{dX^2} + X^{\frac{1}{2}}\frac{dY_0}{dX} = 0$, $Y_0(0) = 0$

$$\Rightarrow \quad \frac{dY_0}{dX} = Ce^{-\frac{2}{3}X^{3/2}} \quad (C \in \mathbb{R})$$

$$\Rightarrow \quad \underline{\underline{Y_0 = C\int_0^X e^{-\frac{2}{3}t^{3/2}}\,dt}}$$

□

(d) (l.t.o.) $= e^{2(1-x^{1/2})}$

$\Rightarrow$ (l.t.o.) in inner variables $= e^{2(1-(\varepsilon^{2/3}X)^{1/2})}$

$= e^2 e^{-\varepsilon^{1/3}X^{1/2}}$

$\sim e^2$

$\Rightarrow$ (l.t.i.)(l.t.o.) $= e^2$

(l.t.i.) $= C\int_0^X e^{-\frac{2}{3}t^{3/2}}\,dt$

$\Rightarrow$ (l.t.i.) in outer variables $= C\int_0^{x/\varepsilon^{2/3}} e^{-\frac{2}{3}t^{3/2}}\,dt$

$\sim C\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt$

$\Rightarrow$ (l.t.o.)(l.t.i.) $= C\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt$

(l.t.i.)(l.t.o.) = (l.t.o.)(l.t.i.) $\Rightarrow$

$$e^2 = C\int_0^\infty e^{-\frac{2}{3}t^{3/2}}\,dt$$

$$= C\left(\tfrac{2}{3}\right)^{1/3}\int_0^\infty s^{2/3-1}e^{-s}\,ds \qquad \left(s = \tfrac{2}{3}t^{3/2},\ \tfrac{ds}{dt} = t^{1/2}\right)$$

$$= C\left(\tfrac{2}{3}\right)^{1/3}\Gamma\left(\tfrac{2}{3}\right) \Rightarrow C = \frac{e^2}{\left(\frac{2}{3}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)}$$

Q4(a) $\varepsilon y'' + yy' - y = 0$ for $0 < x < 1$, with $y(0) = 1$, $y(1) = 3$.

<u>Outer</u>: $y \sim y_0(x) + \varepsilon y_1(x) + \ldots$ as $\varepsilon \to 0^+$ with $x = O(1)$

$O(\varepsilon^0)$: $y_0 y_0' - y_0 = 0$ for $0 < x < 1$

No BL at $x = 1 \Rightarrow$ apply BC $y_0(1) = 3 \Rightarrow$ $\underline{y_0 = x + 2}$

<u>Inner</u>: $x = \delta(\varepsilon)X$, $y = Y(X)$ with $\delta \to 0$, $X = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \frac{\varepsilon}{\delta^2}\frac{d^2Y}{dX^2} + \frac{1}{\delta}Y\frac{dY}{dX} - Y = 0$$

3rd term << 2nd term, so balance 1st and 2nd term by setting $\delta = \varepsilon$ and expand $Y \sim Y_0(X) + \varepsilon Y_1(X) + \ldots$ as $\varepsilon \to 0^+$ with $X = O(1)$

$O(\varepsilon^0)$: $\frac{d^2Y_0}{dX^2} + Y_0\frac{dY_0}{dX} = 0 \Rightarrow \frac{dY_0}{dX} + \frac{1}{2}Y_0^2 = \frac{1}{2}B_1$ $(B_1 \in \mathbb{R})$, with $Y_0(0) = 1$

$B_1 = -w^2$ $(w > 0) \Rightarrow Y_0 = w\tan\frac{w}{2}(X_0 - X)$, $1 = w\tan\frac{wX_0}{2}$ $(X_0 \in \mathbb{R})$

$\Rightarrow$ cannot match without avoiding a singularity in $Y_0(X)$

$B_1 = 0 \Rightarrow Y_0 = \frac{1}{1 + X/2} \Rightarrow$ cannot match $\because Y_0(+\infty) = 1 \neq 2 = y_0(0^+)$

$B_1 = +w^2$ $(w > 0) \Rightarrow Y_0 = w\tanh\frac{w}{2}(X - X_0)$, $1 = w\tanh\left(\frac{-wX_0}{2}\right)$ $(X_0 \in \mathbb{R})$

$\Rightarrow Y_0(+\infty) = w$ and can match with outer.

Hence, need $B_1 = w^2 > 0$ to match, as follows.

Matching

$$(\text{l.t.o.}) = x+2$$
$$\Rightarrow (\text{l.t.o.}) \text{ in inner variables} = \varepsilon X + 2$$
$$\Rightarrow (\text{l.t.i.})(\text{l.t.o.}) = 2$$

$$(\text{l.t.i.}) = w \tanh \frac{w}{2}(X - X_0)$$
$$\Rightarrow (\text{l.t.i.}) \text{ in outer variables} = w \tanh \frac{w}{2}\left(\frac{x}{\varepsilon} - X_0\right) \sim w \text{ as } \varepsilon \to 0 \quad (w>0,\ x>0)$$
$$\Rightarrow (\text{l.t.o.})(\text{l.t.i.}) = w$$

from BC at x=0

$$(\text{l.t.i.})(\text{l.t.o.}) = (\text{l.t.o.})(\text{l.t.i.}) \Rightarrow \underline{w = 2,\ \boxed{X_0 = -\tanh^{-1}\left(\tfrac{1}{2}\right)}}$$

(b) $\varepsilon y'' + yy' - y = 0$ for $0 < x < 1$, with $y(0) = -\frac{3}{4}$, $y(1) = \frac{5}{4}$.

LH outer: $y \sim y_0^L(x)$ as $\varepsilon \to 0$ with $0 < x < x_0$

$$\Rightarrow y_0^L y_0^{L\prime} - y_0^L = 0 \text{ in } 0 < x < x_0, \text{ with } y_0^L(0) = -\tfrac{3}{4}$$

$$\Rightarrow \underline{y_0^L(x) = x - \tfrac{3}{4} \quad \text{for } 0 < x < x_0}$$

RH outer: $y \sim y_0^R(x)$ as $\varepsilon \to 0$ with $x_0 < x < 1$

$$\Rightarrow y_0^R y_0^{R\prime} - y_0^R = 0 \text{ in } x_0 < x < 1, \text{ with } y_0^R(1) = \tfrac{5}{4}$$

$$\Rightarrow \underline{y_0^R(x) = x + \tfrac{1}{4} \quad \text{for } x_0 < x < 1}$$

Inner: $x = x_0 + \varepsilon X$, $y = Y(X) \sim Y_0(X)$ as $\varepsilon \to 0$ with $X = O(1)$

$$\underset{(a)}{\Rightarrow} \frac{d^2Y_0}{dX^2} + Y_0 \frac{dY_0}{dX} = 0 \quad \text{for } -\infty < X < \infty$$

$$\Rightarrow \frac{dY_0}{dX} + \frac{1}{2}Y_0^2 = \frac{1}{2}w^2 > 0 \text{ to avoid a singularity at finite } X \text{ (with } w>0 \text{ wlog)}$$

$$\Rightarrow \quad Y_0(X) = \omega\left(\frac{Be^{\omega X}-1}{Be^{\omega X}+1}\right) \qquad (B \in \mathbb{R})$$

Matching

Same method as in (a) $\Rightarrow y_0^L(x_0^-) = Y_0(-\infty),\ y_0^R(x_0^+) = Y_0(\infty)$

$$\Rightarrow x_0 - \tfrac{3}{4} = -\omega, \quad x_0 + \tfrac{1}{4} = \omega$$

$$\Rightarrow \omega = \tfrac{1}{2}, \quad x_0 = \tfrac{1}{4}$$

NB: B left undetermined at $O(\varepsilon^n)$ $\forall n \in \mathbb{N}_0$! Can pin it down with a WKB analysis.

Q5 $y'' + \varepsilon y' = 0$ for $0 < x < L$, with $y(0) = 0$, $y(L) = 1$.

(a) $y \sim y_0(x) + \varepsilon y_1(x) + \ldots$ as $\varepsilon \to 0^+$ with $x = O(1)$.

$O(\varepsilon^0)$: $y_0'' = 0$ for $0 < x < L$, with $y_0(0) = 0$, $y_0(L) = 1$.

$$\Rightarrow y_0 = x/L$$

$O(\varepsilon^1)$: $y_1'' + y_0' = 0$ for $0 < x < L$, with $y_1(0) = 0$, $y_1(L) = 0$

$$\Rightarrow y_1'' = -1/L$$

$$\Rightarrow y_1 = \frac{x(L-x)}{2L}$$

Hence, $y \sim \dfrac{x}{L} + \varepsilon\dfrac{x(L-x)}{2L} + \ldots$ as $\varepsilon \to 0^+$ with $L = O(1)$ $\square$

(b) This gives $y'(0) \sim \frac{1}{L} + \frac{\varepsilon}{2}$ as $\varepsilon \to 0^+$ with $L = O(1)$, so this expansion is not valid for $L \gg 1/\varepsilon$ and hence in the limit $L \to \infty$. Expansion for $y'(0)$ nonuniform when $\varepsilon L = O(1)$ as $\varepsilon \to 0^+$, corresponding to the distinguished limit in which $L = \lambda/\varepsilon$, $\lambda = O(1)$ as $\varepsilon \to 0^+$. Scaling $\boxed{x = X/\varepsilon,\ y = Y(X)}$ gives $Y'' + Y' = 0$ for $0 < X < \lambda$, with $Y(0) = 0$, $Y(\lambda) = 1$, so that

$$Y(X) = \frac{1-e^{-X}}{1-e^{-\lambda}} \Rightarrow Y'(0) = \frac{1}{1-e^{-\lambda}} \to 1 \text{ as } \lambda \to \infty$$ in agreement with $L \to \infty$ limit of exact solution.

Q6(a) $\varepsilon \nabla^2 u = u$ in $r < 1$, with $u = 1$ on $r = 1$.

<u>Outer</u> : $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $1-r = O(1)$.

$O(\varepsilon^0)$: $u_0 = 0$

$O(\varepsilon^1)$: $u_1 = \nabla^2 u_0 = 0$

$O(\varepsilon^2)$: $u_2 = \nabla^2 u_1 = 0$

small 'oh'

Induction $\Rightarrow$ $\underline{u = o(\varepsilon^n) \ \forall n \in \mathbb{N} \text{ as } \varepsilon \to 0^+}$ □

<u>Inner</u> : $u(r,\theta) = U(R,\theta)$, $r = 1 - \delta(\varepsilon)R$, with $\delta \to 0$, $R = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2} U_{RR} - \frac{\varepsilon}{\delta(1-\delta R)} U_R + \frac{\varepsilon}{(1-\delta R)^2} U_{\theta\theta} - U = 0$$

Balance 1^{st} and 4^{th} term by setting $\delta = \varepsilon^{1/2}$ to obtain

$$U_{RR} - \frac{\varepsilon^{1/2}}{(1-\varepsilon^{1/2}R)} U_R + \frac{\varepsilon}{(1-\varepsilon^{1/2}R)^2} U_{\theta\theta} - U = 0$$

$U \sim U_0(R,\theta) + \varepsilon^{1/2} U_1(R,\theta) + \dots$ as $\varepsilon \to 0^+$ with $R = O(1)$.

$O(\varepsilon^0)$: $U_{0RR} - U_0 = 0$ in $R > 0$, with $U_0 = 1$ on $R = 0$

$$\Rightarrow \quad U_0 = A e^R + (1-A) e^{-R} \quad (A \in \mathbb{R})$$

<u>Matching</u> : (l.t.o.) $= 0$ $\Rightarrow$ (l.t.i.)(l.t.o.) $= 0$

$\Rightarrow$ (l.t.o.)(l.t.i.) $= 0$

$\Rightarrow$ $U_0 \to 0$ as $R \to \infty$

$\Rightarrow$ $A = 0$

Hence, $\underline{u = e^{-R} + O(\varepsilon^{1/2}) \text{ as } \varepsilon \to 0^+ \text{ with } \varepsilon^{1/2}(1-r) = R = O(1)}$

□

Given exact solution $u = I_0(r/\sqrt{\varepsilon})/I_0(1/\sqrt{\varepsilon})$, where

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cos(ix\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi e^{-i(ix\sin\theta)} + e^{i(ix\sin\theta)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^\pi e^{x\sin\theta} + e^{-x\sin\theta}\,d\theta$$

$$\sim \frac{1}{2\pi}\int_0^\pi e^{x\sin\theta}\,d\theta$$ as $x \to \infty$ $\because$ 1st integral is exponentially dominant $\because \sin\theta > 0$ for $0 < \theta < \pi$.

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x[1-\frac{1}{2}(\theta-\frac{\pi}{2})^2+\dots]}\,d\theta$$ via Laplace's method $\because \phi(\theta) = \sin\theta$ has max at $\theta = \pi/2$.

$$\sim \frac{e^x}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{xs^2}{2}}\,ds \qquad (\theta - \tfrac{\pi}{2} = s)$$

$$= \frac{e^x}{\sqrt{2\pi}}\sqrt{\frac{2}{x}}\underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{=\sqrt{\pi}} \qquad (s = \sqrt{\tfrac{2}{x}}\,t)$$

$$\Rightarrow I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty.$$

Thus,

$$u \sim \frac{1}{\sqrt{r}}e^{-(1-r)/\sqrt{\varepsilon}}$$ as $\varepsilon \to 0^+$ with $r = O(1)$, $1-r = O(1)$;

$$u \sim \frac{\sqrt{2\pi}\,e^{-1/\sqrt{\varepsilon}}}{\varepsilon^{1/4}} I_0(\rho)$$ as $\varepsilon \to 0^+$ with $\rho = \varepsilon^{-1/2}r = O(1)$;

$$u \sim \frac{1}{\sqrt{1-\varepsilon^{1/2}R}}e^{-R} = e^{-R} + O(\varepsilon^{1/2})$$ as $\varepsilon \to 0^+$ with $R = \varepsilon^{-1/2}(1-r) = O(1)$,

in agreement with formal BL theory. $\square$

(6) $\varepsilon(u_{xx} + u_{yy}) = u_x$ in $y > 0$;
$u = 1$ on $y = 0, x > 0$;
$u_y = 0$ on $y = 0, x < 0$;
$u \to 0$ as $r \to \infty$.

<u>Outer</u>: $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $x, y = O(1)$

$O(\varepsilon^0)$: $u_{0x} = 0$ with $u_0 = 0$ at ∞ $\Rightarrow u_0 = 0$
$O(\varepsilon^1)$: $u_{1x} = 0$ with $u_1 = 0$ at ∞ $\Rightarrow u_1 = 0$

small 'oh'

Induction $\Rightarrow$ $\underline{u = o(\varepsilon^n) \ \forall n \in \mathbb{N} \text{ as } \varepsilon \to 0^+ \text{ with } x, y = O(1)}$ $\square$

<u>Inner</u>: $u(x,y) = U(x,Y)$, $y = \delta(\varepsilon)Y$, with $\delta \to 0$, $Y = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad \varepsilon U_{xx} + \frac{\varepsilon}{\delta^2} U_{YY} - U_x = 0$$

Balance 2nd and 3rd terms by setting $\delta = \varepsilon^{1/2}$ to obtain

$$\varepsilon U_{xx} + U_{YY} - U_x = 0$$

$U \sim U_0(x,Y) + \varepsilon U_1(x,Y) + \dots$ as $\varepsilon \to 0^+$ with $Y = O(1)$.

$O(\varepsilon^0)$: $U_{0YY} - U_{0x} = 0$ in $Y > 0, x > 0$.

BC: $U_0(x, 0) = 1$ for $x > 0$.

<u>Matching</u>: $(\text{l.t.o.}) = 0 \Rightarrow (\text{l.t.i.})(\text{l.t.o.}) = 0$
$\Rightarrow (\text{l.t.o.})(\text{l.t.i.}) = 0$
$\Rightarrow U_0 \to 0$ as $Y \to \infty$ for $x > 0$.

Seek similarity solution $U_0 = f(\eta)$, $\eta = Y/\sqrt{x}$

$$\Rightarrow \quad \eta_x = -\frac{\eta}{2x}$$

$$\eta_Y = \frac{1}{x^{1/2}}$$

$$U_{0x} = f'(\eta)\eta_x = -\frac{\eta f'(\eta)}{2x}$$

$$U_{0YY} = f''(\eta)(\eta_Y)^2 = \frac{f''(\eta)}{x}$$

PDE $\Rightarrow \quad f'' + \frac{1}{2}\eta f' = 0 \quad$ for $\eta > 0$.

BCs : $U_0 = 1$ on $Y = 0, x > 0 \quad \Rightarrow f(0) = 1$
$U_0 \to 0$ as $Y \to \infty, x > 0 \quad \Rightarrow f(\infty) = 0$

Now, $\frac{f''}{f'} = -\frac{\eta}{2} \Rightarrow \ln|f'| = c_1 - \frac{\eta^2}{4} \quad (c_1 \in \mathbb{R})$

$$\Rightarrow \quad f' = c_2 e^{-\eta^2/4} \quad (|c_2| = e^{c_1})$$

$$\Rightarrow \quad f(\eta) = c_1 - c_2 \int_\eta^\infty e^{-s^2/4}\, ds$$

erfc(p) = (2/sqrt(pi)) \int^\infty_p exp(-s^2) ds

$$= c_1 - 2c_2 \int_{\eta/2}^\infty e^{-t^2}\, dt \quad (s = 2t)$$

$$= c_1 - 2c_2\, \mathrm{erfc}\left(\frac{\eta}{2}\right)$$ WLOG, Relabelling c_2

$f(\infty) = 0 \Rightarrow 0 = c_1 - 0$
$f(0) = 1 \Rightarrow 1 = -2c_2\, \mathrm{erfc}(0) = -2c_2$

Hence, $f(\eta) = \mathrm{erfc}\left(\frac{\eta}{2}\right) \Rightarrow U = \mathrm{erfc}\left(\frac{Y}{2\sqrt{x}}\right) + O(\varepsilon)$ as $\varepsilon \to 0^+$
with $Y = \varepsilon^{-1/2}y = O(1)$
and $x = O(1)$ $\square$

Neither approximation holds for $X = \varepsilon^{-1}x = O(1), Y = \varepsilon^{-1}y = O(1)$
$\Rightarrow u_{XX} + u_{YY} = u_X$ in $Y > 0$.